On Multiple Delamination Buckling of Composite Structures

Shapour Moradi[1] and Farid Taheri[2]
[1]Department of Mechanical Engineering, Shaheed-Chamran University, Ahvaaz, Iran
[2]Department of Civil Engineering, Dalhousie University
P.O. Box 1000, Halifax Nova Scotia, Canada B3J 2X4

SUMMARY: In this study the buckling response of a one-dimensional composite beam having multiple delaminations and subjected to axial compressive load is modelled by the method of differential quadrature. The delaminations are considered to be in arbitrary locations through-the-width, and along the span of the beam. A beam theory with shear deformation is used to formulate the problem. To model the problem, the beam is divided into a series of regions bounded by the delaminated segments. Using DQM, the system of equilibrium equations and the boundary conditions are transformed into a system of linear algebraic eigenvalue equations that are solved by a standard eigen-solver. The influences of several parameters that affect the buckling strength of such laminates are investigated. The numerical results obtained from a series of case studies are compared with those of the finite elements method and other published results. These comparisons confirm the efficiency and accuracy of the method.

KEYWORDS: Composites, beams, buckling, differential quadrature, multiple-delaminations, shear-effect, numerical modeling.

INTRODUCTION

The definite advantages of laminated fibre-reinforced composites (i.e., their low weight, high specific strength and stiffness, etc.), is well recognized but, they suffer from a major problem, namely their weak strength in the through-the-thickness direction, mostly due to low cohesive interlaminar strength. This deficiency becomes more pronounced if the composite laminate also contains delaminations. Delaminations in composites may develop during manufacturing, because of the imperfections and/or faulty procedures, or during service, by impact of an external object. This can significantly reduce the compressive strength and stiffness of the laminate and thereby, reduce the buckling load of the laminate when subjected to a compressive load. As a consequence the delaminated region further grows, often causing the complete failure of the part.

The complex nature of delamination, and its associated formulation often requires numerical treatment. In the past two decades, several workers have employed effective and popular numerical methods such as the Finite Difference Method (FDM), the Finite Element Method (FEM) to solve delamination problems in composites. FDM is one of the simplest numerical methods (both in terms of its formulation and programming effort). To obtain an accurate result, however, considerable effort is required for representing (discretizing) the domain by a large number of grid points. FEM on the other hand requires more skill and effort for algorithm development and implementation. Other numerical methods such as the Rayleigh-Ritz have also

been used for solving such problems. Thus, the development of new methods from the standpoint of numerical accuracy, ease of formulation and computational efficiency is still of prime interest.

A relatively new numerical technique is the differential quadrature method (DQM). Bellman et.al. [1] introduced the method in the early seventies for solving linear and nonlinear partial differential equations. Differential quadrature has been shown to perform extremely well in solving initial and boundary value problems (Bert and Malik [2]). Moradi and Taheri [3,4] used the method to analyze the delamination buckling of laminated beams. Their investigation indicated that DQM could be considered as an efficient and a reliable numerical tool in treating the problem of delamination buckling.

This work is an extension of our work presented earlier; in this work we examine the validity and applicability of the DQM to analyze the response of a composite beam hosting several delaminated regions, subjected to an axial compressive load. The delaminations can be at any arbitrary location through-the-width, and along the span of the beams. The formulation adopted a beam theory that included the effect of shear deformation. Within the scope of the proposed methodology, the beam is divided into a series of regions bounded by the delaminated segments. Then, the DQM is applied to transform the system of equilibrium equations and the boundary conditions into a system of linear algebraic eigenvalue equations that can be solved by any standard eigen-solver. For the validation of the method, we considered the influences of three parameters, namely: the lengths, the position and the number of delaminations, on the buckling response. The integrity of the results obtained through DQM was verified by the results obtained from the finite element method and other published results. These comparisons confirm the efficiency and accuracy of the method.

DIFFERENTIAL QUADRATURE METHOD

DQM is based on the weighted linear sum of function values as an approximation to the derivatives of that function. Bellman et. al. [1] stated that the derivative of a function with respect to a space variable could be approximated by a weighted linear combination of function values evaluated at some intermediate points in the domain of that variable. Mathematically, the nth order derivative of the function $F(x)$ at an intermediate discrete point x_i can be approximated by the weighted linear sum of the function values as:

$$\frac{d^n F(x_i)}{d\,x^n} = \sum_{j=1}^{N} c_{ij}^{(n)} F(x_j) \qquad \begin{array}{c} i = 1,\ldots,N \\ n = 1,\ldots,N-1 \end{array} \tag{1}$$

Here, the domain is divided into N discrete regions, and $c_{ij}^{(n)}$ are the weighting coefficients of the nth derivative, where $n \le N-1$. The weighting coefficients can be determined through the following recurrence formulae [1]:

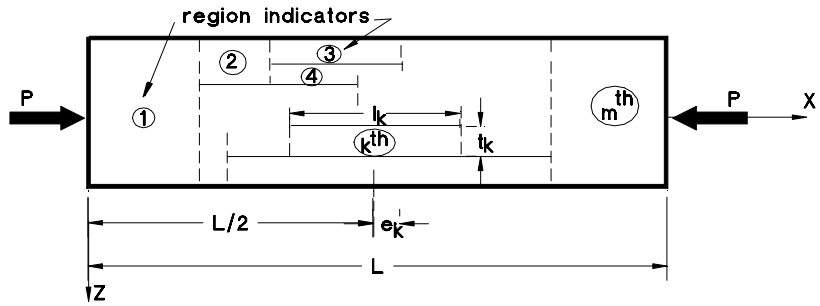


Figure 1: Geometry of a beam with multiple delaminations.

$$c_{ij}^{(1)} = \frac{\prod(x_i)}{(x_i - x_j)\cdot\prod(x_j)} \qquad i, j = 1,\ldots,N \quad \text{and} \quad j \neq i$$

$$c_{ij}^{(r)} = r\left[c_{ii}^{(r-1)}\cdot c_{ij}^{(1)} - \frac{c_{ij}^{(r-1)}}{x_i - x_j}\right] \qquad 2 \le r \le N-1 \tag{2}$$

$$c_{ii}^{(m)} = -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}^{(m)} \qquad m = 1,\ldots,N-1$$

where

$$\prod(x_i) = \prod_{\substack{j=1\\ j\neq i}}^{N}(x_i - x_j) \tag{3}$$

The above relations are not restricted to the choice of sampling points; they are simple and easy to implement, hence, reducing the computational effort, significantly. A simple, and at the same time a more efficient mean for selecting the sampling points is achieved by selecting the roots of the shifted Chebyshev polynomials in the [0,1] domain, as represented by:

$$x_i = \frac{1}{2}\left(1 - \cos\frac{2i-1}{2N}\pi\right) \tag{4}$$

For a detailed discussion on the evaluation and selection of the weighting coefficients and the discrete points for DQM, see Bert and Malik [2], and Moradi and Taheri [5]. In order to apply DQM, first we have to establish the governing differential equations of the beam hosting multiple delaminations.

BUCKLING FORMULATION FOR A LAMINATED COMPOSITE BEAM WITH MULTIPLE DELAMINATION

The side view of a composite beam hosting several through-the-width delaminations is illustrated in Figure 1. The delaminations are of arbitrary lengths and also positioned in arbitrary locations along the beam span. The sections before and after delaminated regions, where the plate is intact are named the “base” laminates, and the delaminated regions are referred to as the “sublaminates”. The delaminations divide the beam into m geometrically continuous regions (identified in the figure by doted lines). t_k and l_k are the thickness and length of each region, respectively. The position of each region along the beam span is defined by e_k. The beam is under an axial compressive load.

Using the first order shear deformation theory to treat each region "k" as a separate beam, the governing equilibrium equations corresponding to each region can be represented by:

$$D_k \psi_{k,xx} - k_s A_{55}^{(k)} \left(\psi_k + w_{k,x} \right) = 0 \qquad (5a)$$

$$k=1,2,\ldots,m$$

$$k_s A_{55}^{(k)} \left(\psi_k + w_{k,x} \right)_{,x} - P_k w_{k,xx} = 0 \qquad (5b)$$

where w_k, ψ_k and P_k are the transverse deflection, transverse rotation and the resulting compressive force of the kth region, $A_{55}^{(k)}$ and D_k are the stiffness of each region defined by:

$$A_{55}^{(k)} = Q_{55}^{(k)} . \int_{-t_k/2}^{t_k/2} dz \qquad D_k = D_{11}^{(18)} - \frac{B_{11}^{(k)}}{A_{11}^{(k)}} \qquad (6)$$

The boundary conditions for a multiply delaminated beam with clamped ends consist of the in-plane, transverse, continuity and compatibility conditions given by:

at x=0 ➠ $P_1 = -P$ $\quad w_1 = \psi_1 = 0$ while at x=L ➠ $P_m = -P$ $\quad w_m = \psi_m = 0$ (7a)

At the delaminated edges, the geometric boundary conditions consist of vertical deformation and rotation compatibility equations as:

$$w_i = w_j = \ldots = w_k \qquad \text{and} \qquad \psi_i = \psi_j = \ldots = \psi_k \qquad (7b)$$

where $i,j,...,k$ are the region numbers surrounding a crack tip.

The other boundary conditions at delamination fronts consist of the in-plane force, shear force and moment equations as follow:

$$P_i - P_j - \cdots - P_k = 0 \qquad (8c)$$

$$Q_i - Q_j - \cdots - Q_k - \left(P_i w_{i,x} - P_j w_{j,x} - \cdots - P_k w_{k,x} \right) = 0 \qquad (8d)$$

$$M_i - M_j - \cdots - M_k + P_i z_i + P_j z_j + \cdots + P_k z_k = 0 \qquad (8e)$$

where z is the distance between the mid-plane of each region and the appropriate moment point. Shear force Q and moment M are defined for each region, by the following equations, respectively:

$$Q_k = k_s G t_k \left(\psi_k + w_{k,x} \right) \qquad (9a)$$

$$M_k = D_k \psi_{k,x} \qquad (9b)$$

In order to solve for the buckling load, the axial compressive load P in equations (8c,d,e) should be eliminated. To do this, one can combine the strain compatibility equation for the regions

between delaminations given by Eq 10 and the kinematics relation in 8(b), to obtain the compatibility relations as given in Eq1 11(a) and 10.

$$P_k = \frac{A_{55}^{(k)}}{A_{55}^{(1)}} P \tag{10}$$

$$D_1\psi_{1,x} - D_2\psi_{2,x} - D_3\psi_{3,x} + \frac{A_{11}^{(2)}A_{11}^{(3)}}{A_{11}^{(1)}}\left(\frac{B_{11}^{(3)}}{A_{11}^{(3)}} - \frac{B_{11}^{(2)}}{A_{11}^{(2)}} + T\right)\cdot\left(\frac{B_{11}^{(3)}}{A_{11}^{(3)}} - \frac{B_{11}^{(2)}}{A_{11}^{(2)}} + \frac{T}{2}\right)\frac{\psi(l)}{a} = 0 \tag{11a}$$

$$D_2\psi_2 + D_3\psi_3 - D_4\psi_4 - \frac{A_{11}^{(2)}A_{11}^{(3)}}{A_{11}^{(1)}}\left(\frac{B_{11}^{(3)}}{A_{11}^{(3)}} - \frac{B_{11}^{(2)}}{A_{11}^{(2)}} + T\right)\cdot\left(\frac{B_{11}^{(3)}}{A_{11}^{(3)}} - \frac{B_{11}^{(2)}}{A_{11}^{(2)}} + \frac{T}{2}\right)\frac{\psi(l)}{a} = 0 \tag{11b}$$

Applying the differential quadrature technique to the differential equilibrium equations of all *k* regions divided by the discrete section, results in a system of discrete algebraic equations as:

$$\frac{D_k}{l_k^{\,2}}\sum_{j=1}^{N_k} C_{ijk}^{(2)}\Psi_{jk} - k_s A_{55}^{(k)}\left(\delta_{ij}\Psi_{jk} + \frac{1}{l_k}\sum_{j=1}^{N_k} C_{ijk}^{(1)} W_{jk}\right) = 0 \qquad \begin{matrix} i = 1,\ldots,N \\ k = 1,\ldots,m \end{matrix} \tag{12a}$$

$$\sum_{j=1}^{N_k} C_{ijk}^{(1)}\Psi_{jk} + \frac{1}{l_k}\sum_{j=1}^{N_k} C_{ijk}^{(2)} W_{jk} = \left(\frac{P}{k_s A_{55}^{(1)} l_k}\right)\sum_{j=1}^{N_k} C_{ijk}^{(2)} W_{jk} \qquad \begin{matrix} i = 1,\ldots,N \\ k = 1,\ldots,m \end{matrix} \tag{12b}$$

Applying the differential quadrature to the boundary conditions, together with the above equations produce a system of eigenvalue equations. This system can be solved by any standard eigenvalue solver to give the delamination buckling loads and corresponding modal shapes. For details of the conversion of the DQM equation to an eigenvalue type equation, see Moradi and Taheri {5}.

CASE STUDIES FOR BUCKLING OF A BEAM HAVING MULTIPLE DELAMINATIONS

To validate the integrity of the outlined methodology, we analyzed several case studies. As a first case, we considered a specially orthotropic beam having equal length delaminations. The delaminations are through-the-width and could be situated anywhere along the longitudinal span of the beam.

Effect of the number of delaminations on the buckling load

Figure 2 shows the effect of number of delaminations on the buckling strength of the beam. This case study was also analyzed by Suemasu [5], for which he used Rayleigh-Ritz approximation to solve for the buckling response. The material properties are given in Table 1. The length and thickness of the beam were taken as 160 mm and 3.8 mm, respectively. The effect of the shear deformation is accounted by the shear deformation factor *s=5/6*. The number of delaminations

is represented by *N*. They divide the clamped-clamped beam into (*N+1*) regions with equal thickness.

Table 1: Material properties used by Suemasu [5].

E_1	20.2
E_2	21.0 GPa
E_3	10.0 GPa
υ_{12}	0.16
υ_{13}	0.30
G_{12}	4.15 GPa
G_{13}	4.0 GPa

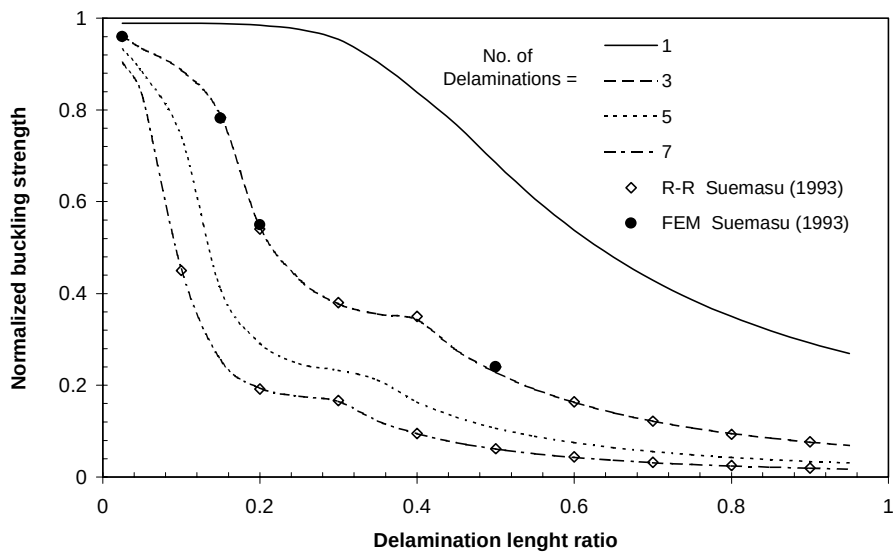


Figure 2: Normalized buckling load vs. delamination length ratio for multiple delaminations.

In this graph the buckling strengths are normalized with respect to the Euler buckling strength of an intact beam. As shown in the figure, the DQM results are in very good agreement with those of the finite element and Rayleigh-Ritz calculations given by Suemasu [5]. Increasing the number of delaminations (and therefore reducing the thickness of delaminations, reduces the buckling strength. Also, the sharp changes in curvature of all the curves are due to the transformation in buckling modes. The first and the last regions of the curves are govern by symmetric modes while in the mid-region, the antisymmetric modes are the dominant modal shapes.

Effect of the longitudinal position of delaminations

The influence of the longitudinal position of delaminations on the buckling strength is shown in figures 3 and 4, for different number of delaminations. As expected, the maximum buckling strengths are attained when the delaminations are in the center span of the beam. Changing the delamination position results in a decrease in the buckling strength. This effect is more obvious in the cases where delaminations are relatively small. Also, increasing the number of

delaminations makes the beam less sensitive to the change in the longitudinal position of the delaminations. This is more noticeable when the normalized buckling strengths are plotted versus different delamination positions for different number of delaminations. (Figure 4).

Effect of the length of delaminations

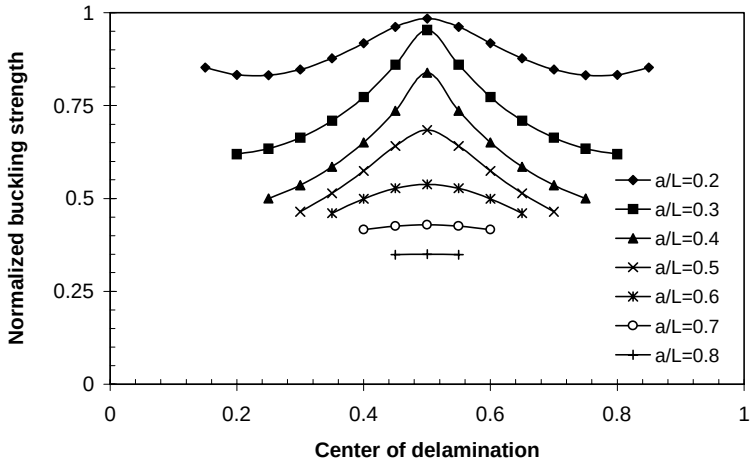
The effect of the variation in the length of delaminations on the buckling strength is shown in Figure 5. Here the beam was assumed to be supported clamped-clamped having two delaminations with different lengths, located symmetrically at the beams mid-spans. Moreover, the delaminations are positioned symmetrically with respect to the mid-plane of the beams. While the length of the upper delaminations in these beams varies, but the length of the lower ones are fixed (a/L=0.3,0.4). The non-dimensional buckling strength as a function of the non-dimensional delamination length of the upper delaminations for the case of thin (t/T=0.125) and thick (t/T=0.3) delaminations are presented in the figure. The results are compared with those obtained from the finite element analysis conducted using the commercial package NISA [7]. Also, the results for the beams with a single delamination are shown in the same figure. The curves indicates that as long as the length of the lower delamination is less than the length of the upper delamination (i.e. $a_2 > a_1$), the buckling strength of the beam is governed by the length of the upper delamination. This response is consistent with the case where the beam hosts a single delamination.

CONCLUSIONS

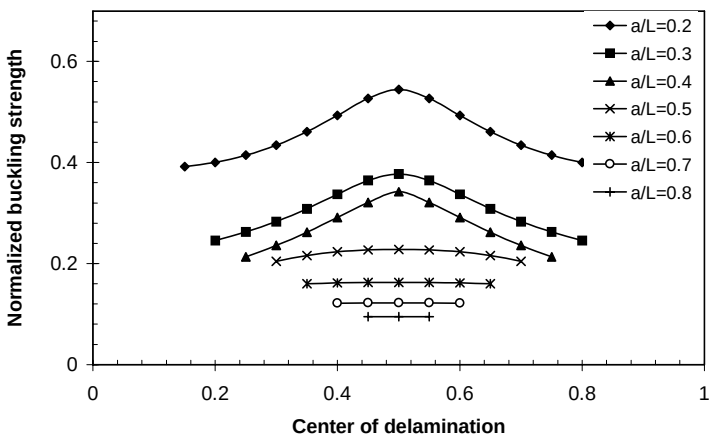
The buckling response of a one-dimensional beam-plate having multiple across-the-width delaminations that were located in an arbitrary locations (through-the-thickness and along the length of the beam-plate), was characterized by the differential quadrature method (DQM). A beam theory with shear deformation was used to formulate the problem. Using DQM the system of the differential equations was transformed into a system of linear algebraic eigenvalue equations, solvable by any standard eigen-solver.
Based on the foregoing analysis the following conclusions are made:

1. The delamination buckling strength is very sensitive to the longitudinal position of delaminations. The sensitivity is reduced however, as the number of delaminations is increased.
2. The delaminations lengths also influence the buckling strength of the beams. This phenomenon is consistent, both for thin delaminations, as well as for thick delaminations.
3. The DQM can provide excellent results in comparison to the FEM.

In summary, the simplicity, efficiency, and accuracy of DQM suggest that the method can be considered as a suitable tool for characterizing the delamination buckling response of composites.



(a)



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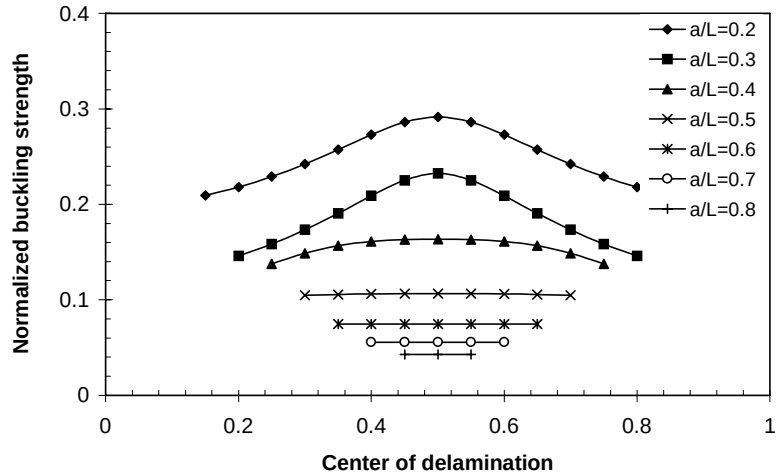

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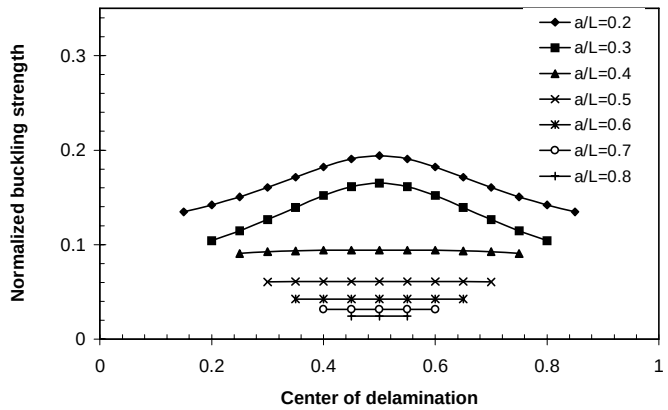

(d)

Figure 3. Effect of the longitudinal position of delaminations on the buckling strength for a beam with (a) a single delamination, (b) three delaminations, (c) five delaminations and (d) seven delamination

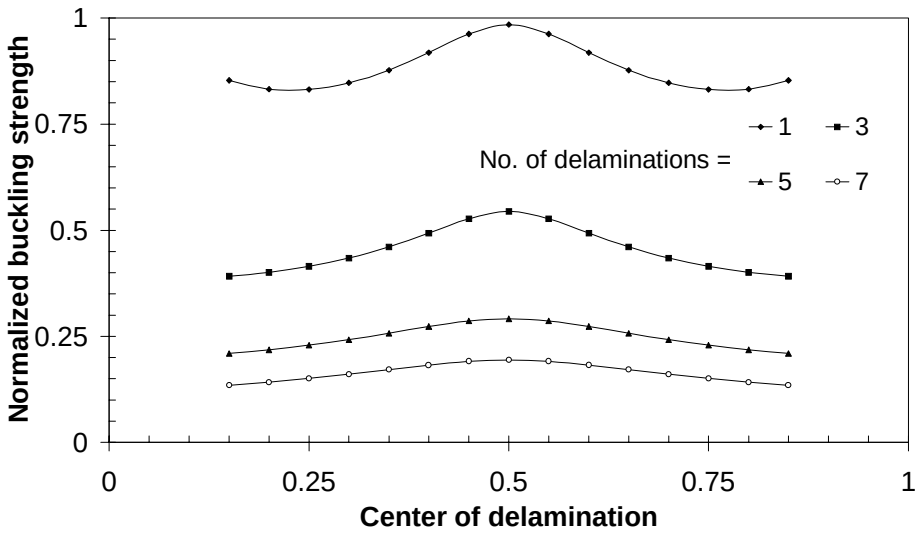

Figure 4. Comparison of the influence of the longitudinal position of delaminations on the buckling strength for a beam with different number of delaminations (a/L=0.2).

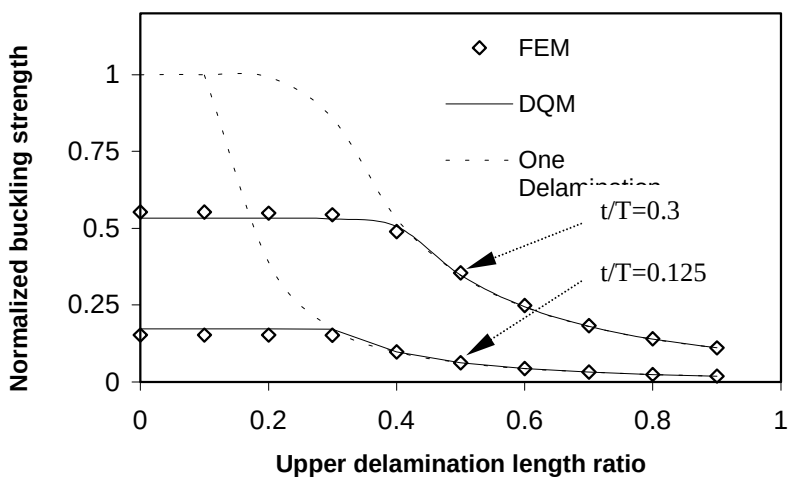

Figure 5. Response of beams hosting multiple delaminations with different lengths.

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