Prediction and Evaluation of Local and Global Optima in Strength Design of Composite Laminates

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SUMMARY: A thicker composite laminate is not necessarily safer than a thinner one of the same relative proportions. Although subjected to the same loadcase, it may even fail! Therefore, the paths followed by strains in laminates upon proportional scaling of their thickness are investigated so as to determine upfront the number and range of safe design thickness allowables. It is shown that one or two such intervals may exist. Their existence and boundaries are predicted by decomposing nominal strains into N-strains and M-strains, i.e. the strain fractions due to in-plane (N) and bending (M) loads respectively. Incidently, this analytical prediction method enables identification and localization of local and global strength optima. An example reveals the inherent consequences on manufacturing tolerances and design safety margins.

KEYWORDS: optimization, strength, scaling, local/global, mechanical properties

1 Introduction

Strength optimization of composite laminates for minimal weight is a complex task. Overall, mechanical failure criteria are best suited for analysis of given structures, rather than for design of yet unknown ones. A scaling criterion, adapted for the latter purpose, was proposed by Manne and Tsai [1]. It provides the exact ratio λ by which the thickness of a reference laminate must be scaled in order to sustain, without failure, all imposed loadsets. This formulation, based on the quadratic Tsai-Wu criterion [2, 3], yields up to four roots. Investigation of the evolution of strains in laminates with thickness scaling revealed that such multiplicity is physically based.

To illustrate this point, let us consider a small slab of isotropic aluminum (Al-6061) and subject it to a combination of in-plane ($N_1 = 2$ MN/m) and bending ($M_1 = -3$ kNm/m) loads. It is found that a minimum thickness of 15.4 mm, and thus a weight of 41.6 g/m^2 is required for strength. As expected by intuition, any higher thickness fulfills the requirement and further increases the factor of safety (Figure 1 – top).

Let us now subject a $[0_{4\lambda}/90_{4\lambda}]$ laminate made of T300/N5208 to the same loadcase and optimize λ for strength. It is found that $\lambda \geq 4.79$ does the job, yielding 4.79 mm in thickness and 7.66 g/m^2. It is however also shown that any λ between 7.24 and 24.0 makes

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Figure 1: Allowable (highlighted) and failing (shaded/crossed out) thickness ranges for an Al-6061 aluminum (top) and a T300/N5208 $[0_{4\lambda}/90_{4\lambda}]$ (bottom) specimen, subject to longitudinal tension (N_1 = 2 MN/m) and bending ($M_1 = -3$ kNm/m).

it fail again, although yielding a much thicker (7.24 to 24.0 mm) and heavier (11.6 to 38.3 g/m^2) laminate of the same relative composition (Figure 1 – bottom). It is further proven that all laminates can only feature either one or two – like in this example – allowable thickness ranges, no less, no more.

Analysis of the strain paths reveals that this behavior is solely related to the strain pattern and is intrinsically independent of the chosen failure criterion.

Our method predicts upfront the actual number of allowable thickness ranges and evaluates their boundaries. The minimum thickness of the lowest range ($\lambda = 4.79$ in our example) represents the global minimum while the minimum of the upper range ($\lambda = 24.0$) constitutes a local optimum, yielding here a five-fold weight.

2 Strength scaling criterion

In a nutshell, our strength scaling criterion provides the thickness scaling factor λ needed to yield a strength ratio R; usually, $R = 1$. Basically, in strength analysis, $\lambda = 1$ and R is calculated with the standard quadratic criterion, whereas in strength design, R is given and λ is calculated with our scaling criterion. We distinguish two types of scaling:

- Global scaling [1] scales the entire laminate by the same magnitude, in which case the thickness of each individual ply is multiplied by this factor λ (Figure 2).

- Selective scaling [4], however, keeps the central or any designated sublaminate unchanged by excluding it from scaling. This is desired for sandwich, thin-wall and other special constructions, but is not further addressed here.

In the original laminate, the off-axis strain vector is customarily defined as

$$\epsilon = \epsilon^0 + zk \tag{1}$$

$$\left\{ \begin{matrix} \epsilon^0 \\ k \end{matrix} \right\} = \begin{pmatrix} \alpha & \beta \\ \beta^T & \delta \end{pmatrix} \left\{ \begin{matrix} N \\ M \end{matrix} \right\} \tag{2}$$

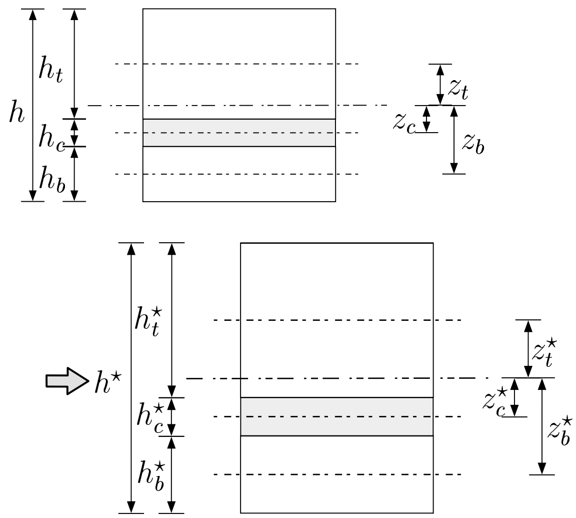

Figure 2: Global thickness scaling of a laminate: $x^\star = \lambda x$.

We rather use decomposition into *N-strains*, ϵ^N, the fraction of ϵ due to the in-plane forces N, and into *M-strains*, ϵ^M, the fraction of ϵ due to the moments M [1]

$$\epsilon = \epsilon^N + \epsilon^M \tag{3}$$
$$\epsilon^N = (\alpha + z\beta^T)N \tag{4}$$
$$\epsilon^M = (\beta + z\delta)M \tag{5}$$

on which the strength scaling criterion is based, for both symmetric and unsymmetric laminates [1]

$$R^2(\epsilon^{M^T}\mathcal{G}\epsilon^M) + 2R^2\lambda(\epsilon^{N^T}\mathcal{G}\epsilon^M) + \lambda^2(R^2\epsilon^{N^T}\mathcal{G}\epsilon^N + RG^T\epsilon^M) + R\lambda^3(G^T\epsilon^N) - \lambda^4 = 0 \tag{6}$$

The roots of this fourth degree polynomial are found by constructing the real upper Hessenberg matrix and by using the QR algorithm to extract them [5]. The physical nature of each of these roots is discussed in Section 4.

3 Effects of thickness scaling on strains

In terms of on-axis *N-strains* and *M-strains*, each strain component scales as [1, 4]

$$\varepsilon_i^\star = \varepsilon_i^{N\star} + \varepsilon_i^{M\star} = \frac{1}{\lambda}\varepsilon_i^N + \frac{1}{\lambda^2}\varepsilon_i^M \tag{7}$$

The thickness scaling factor λ_i^0 such that the strain component $\varepsilon_i^\star$ becomes null is hence equal to

$$\lambda_i^0 = -\frac{\varepsilon_i^M}{\varepsilon_i^N} \tag{8}$$

We call λ_i^0 the *zeroing* thickness scale.

Taking the first derivative of $\varepsilon_i^\star$

$$\frac{\partial\varepsilon_i^\star}{\partial\lambda} = -\frac{1}{\lambda^2}\varepsilon_i^N - \frac{2}{\lambda^3}\varepsilon_i^M \tag{9}$$

yields the thickness scaling factor λ_i^{cr} such that the strain component $\varepsilon_i^\star$ reaches an extremum, other than infinity:

$$\lambda_i^{cr} = -2\frac{\varepsilon_i^M}{\varepsilon_i^N} = 2\lambda_i^0 \tag{10}$$

We call λ_i^{cr} the *critical* thickness scale. The critical strain value is hence equal to

$$\varepsilon_i^{cr} = -\frac{1}{4}\frac{\varepsilon_i^{N\,2}}{\varepsilon_i^M} \tag{11}$$

Taking the second derivative of $\varepsilon_i^\star$

$$\frac{\partial^2 \varepsilon_i^\star}{\partial \lambda^2} = \frac{2}{\lambda^3}\varepsilon_i^N + \frac{6}{\lambda^4}\varepsilon_i^M \tag{12}$$

yields the thickness scaling factor λ_i^{pi} such that the strain component $\varepsilon_i^\star$ reaches its point of inflexion:

$$\lambda_i^{pi} = -3\frac{\varepsilon_i^M}{\varepsilon_i^N} = 3\lambda_i^0 = \frac{3}{2}\lambda_i^{cr} \tag{13}$$

We call λ_i^{pi} the *inflexion* thickness scale. The inflexion strain value is hence equal to

$$\varepsilon_i^{pi} = -\frac{2}{9}\frac{\varepsilon_i^{N\,2}}{\varepsilon_i^M} = \frac{8}{9}\varepsilon_i^{cr} \tag{14}$$

Obviously, the existence of the zeroing, critical, and inflexion strains depends on the relative signs of ε_i^N and ε_i^M. Besides, only positive values of the thickness scaling factor λ make physical sense. If both ε_i^N and ε_i^M are of the same sign, the strain magnitude decreases monotonically from infinity to zero. If the signs are opposite, the strain successively decreases monotonically from infinite magnitude to zero (λ_i^0), reverses sign, peaks in opposite magnitude at λ_i^{cr}, decreases its magnitude asymptotically down to zero whilst going through an inflexion point at λ_i^{pi} (Figures 3 and 4). The different sign combinations of ε_i^N and ε_i^M are analyzed in Table 1.

These results are remarkable in many respects. They are based on pure analysis of strains and are completely independent of any failure theory whatsoever. No assumptions are made, but the standard Kirchhoff's hypotheses (i.e. normals remain straight) [6]. More, the three particular values of λ (Eqs. 8, 10, 13) are in an exact ratio of 1, 2, 3!

Finally, no similarly simple analytical expressions were found for selective scaling. A similar pattern is however expected since one can always find two virtual original layups that yield the same final layup, one by global scaling, the other by selective scaling.

4 Solutions of the strength scaling criterion

4.1 Number of roots

The roots of the strength scaling criterion (Eq. 6) come in either of these combinations:

- One negative real root and three positive real roots ($\lambda_1, \lambda_2, \lambda_3$). The positive roots may also degenerate into a limit subcase: the lowest positive real root (λ_1) and a double positive real root ($\lambda_2 = \lambda_3$).

Table 1: Scaling of strain. Decomposition of strain into its *N-strain* (ε_i^N) and *M-strain* (ε_i^M) components. Sign study around the zeroing (λ_i^0), critical (λ_i^{cr}), and inflexion (λ_i^{pi}) thickness scales.

Case	ε_i^N	ε_i^M	λ_i^0	λ_i^{cr}	λ_i^{pi}	$\varepsilon_i^\star$	$\frac{\partial \varepsilon_i^\star}{\partial \lambda}$	$\frac{\partial \lvert\varepsilon_i^\star\rvert}{\partial \lambda}$	$\frac{\partial^2 \varepsilon_i^\star}{\partial \lambda^2}$
						$\leftarrow \lambda_i^0 \rightarrow$	$\leftarrow \lambda_i^{cr} \rightarrow$	$\leftarrow \lambda_i^0 - \lambda_i^{cr} \rightarrow$	$\leftarrow \lambda_i^{pi} \rightarrow$
A	$+$	$+$				$+\ +\ +$	$-\ -\ -$	$-\ -\ -\ -\ -$	$+\ +\ +$
B	$+$	$-$	$\surd$	$\surd$	$\surd$	$-\ 0\ +$	$+\ 0\ -$	$-\ \mid +\ 0\ -$	$-\ 0\ +$
C	$-$	$+$	$\surd$	$\surd$	$\surd$	$+\ 0\ -$	$-\ 0\ +$	$-\ \mid +\ 0\ -$	$+\ 0\ -$
D	$-$	$-$				$-\ -\ -$	$+\ +\ +$	$-\ -\ -\ -\ -$	$-\ -\ -$

- $\varepsilon_{i_A} = +\varepsilon_i^N + \varepsilon_i^M$
- $\varepsilon_{i_B} = +\varepsilon_i^N - \varepsilon_i^M$
- $\varepsilon_{i_C} = -\varepsilon_i^N + \varepsilon_i^M$
- $\varepsilon_{i_D} = -\varepsilon_i^N - \varepsilon_i^M$

- $\varepsilon_{i_A}^\star = \varepsilon_{i_B}^\star + \dfrac{2}{\lambda^2}|\varepsilon_i^{M\star}|\,, \quad \varepsilon_{i_A}^\star > \varepsilon_{i_B}^\star \quad \forall\lambda$
- $\varepsilon_{i_D}^\star = \varepsilon_{i_C^\star} - \dfrac{2}{\lambda}|\varepsilon_i^{N\star}|\,, \quad \varepsilon_{i_D}^\star < \varepsilon_{i_C}^\star \quad \forall\lambda$

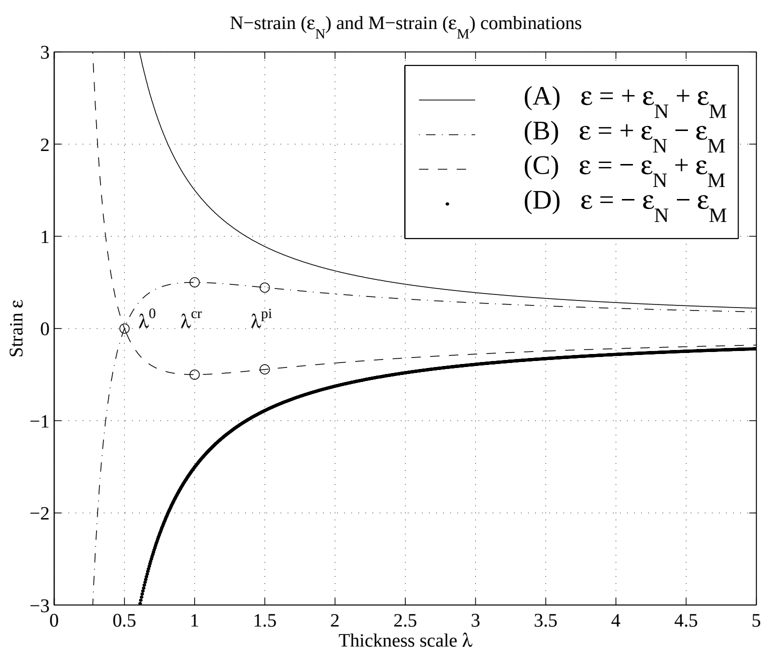

Figure 3: Sign analysis of strains based on combinations of the *N-strains* (ε_i^N) and the *M-strains* (ε_i^M). Four cases (A – D) detailed in Table 1. The three special scales (λ^0, λ^{cr}, and λ^{pi}) of cases B and C are shown with an "*o*" mark.

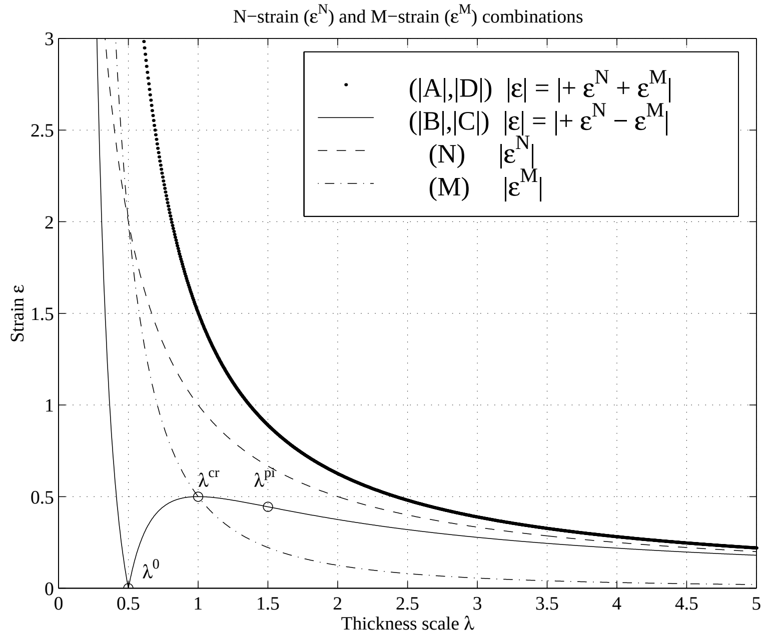

Figure 4: Superposition of the *N-strains* (ε_i^N) and the *M-strains* (ε_i^M). Cases A through D (Table 1) in absolute value. The three special scales (λ^0, λ^{cr}, and λ^{pi}) are shown with an "*o*" mark.

- One negative real root, one positive real root (λ_1), and one pair of complex conjugate roots.

Several interpretations of the three-root case are possible:

- Let us consider the maximum strain criterion. From the strain scaling plots (Figures 3 and 4), we see that the relation between strain intensity and thickness scale is not always single-valued. When the N-strains and M-strains are of opposite sign (cases B and C), we can have three scale roots: $\lambda_1 < \lambda^0$ in one mode (e.g. compression), as well as $\lambda^0 < \lambda_2 \leq \lambda^{cr}$ and $\lambda^{cr} \leq \lambda_3$ in the other mode (e.g. tension). Note that $\lambda_2 = \lambda_3 = \lambda^{cr}$ represents the degenerated double root case. Otherwise, we have only one intersecting scale: $\lambda_1 > 0$ (cases A and D).

- Plotting ϵ_j vs ϵ_i at each ply interface shows the strain paths in function of the thickness scale λ (ϵ_2 vs ϵ_1 in Figure 5). If we superimpose failure envelopes in the same space (Tsai-Wu criterion in this case; maximum strain would be represented by a rectangular envelope), we see that all strain paths are initially (i.e. when $\lambda \ll$) outside of these envelopes; then, one by one, they enter their corresponding envelopes; once inside, the paths either remain there or loop once more outside. Such external peaks correspond to the worst combination of λ_i^{cr} and λ_j^{cr} and lead to failure. If the strain path loop just hits the failure envelope while remaining inside, we have again the degenerated case, with a null safety margin at this particular value.

- If we look at the load reserve margin (R for Tsai-Wu) curves of the laminate (Figure 6), as a function of scaling, we reiterate the same conclusions. We recall that

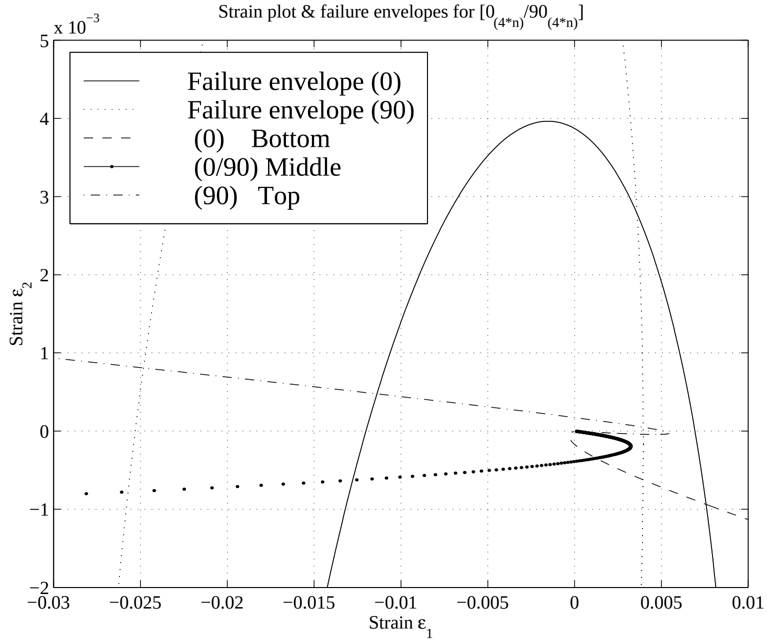

Figure 5: Strain scaling paths and failure envelopes: ϵ_2 vs ϵ_1 for a $[0_{4\lambda}/90_{4\lambda}]$ laminate. Loadcase: $N_1 = 2$ MN/m, $M_1 = -3$ kNm/m.

strains scale globally as follows (Eq. 7):

$$\varepsilon^\star = \frac{1}{\lambda}\varepsilon^N + \frac{1}{\lambda^2}\varepsilon^M \tag{15}$$

Their mathematical limit cases are:

$$\lim_{\lambda\to\pm\infty} \varepsilon^\star = 0 \tag{16}$$

$$\lim_{\lambda\to 0} \varepsilon^\star = \infty \tag{17}$$

On the $R(\lambda)$-curves, this means that $R(\pm\infty) = \infty$ (infinite thickness yields zero strain), that $R(0) < 0$ (zero thickness means infinite strain), and that $R'(0) > 0$ (a little bit of material provides some load carrying capacity, hence an improvement).

4.2 Safe design intervals

From the discussion above and to the contrary of basic intuition, we conclude that one or two thickness scaling intervals exist for safe design. Since at least one positive real root always exists, a solution is mathematically guaranteed, as expected physically.

The following situations are possible:

- One positive real root (λ_1). The laminate fails for $0 \le \lambda < \lambda_1$ and is safe for all $\lambda \ge \lambda_1$. The unique optimum is thus λ_1.

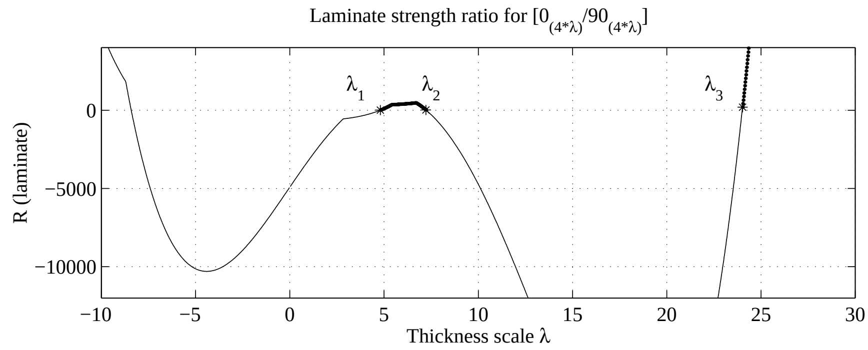

Figure 6: Tsai-Wu failure criterion for a $[0_{4\lambda}/90_{4\lambda}]$ laminate. Strength ratio $R(\lambda)$. Loadcase: $N_1 = 2$ MN/m, $M_1 = -3$ kNm/m. The bold curve segments represent the allowable thickness ranges ($R > 1$).

- Three positive real roots (λ_1, λ_2, and λ_3). The laminate fails for $0 \leq \lambda < \lambda_1$ as well as for $\lambda_2 < \lambda < \lambda_3$. It is safe for $\lambda_1 \leq \lambda \leq \lambda_2$ and for all $\lambda \geq \lambda_3$. The global optimum is thus λ_1, while λ_3 is a larger local optimum.
- The degenerated case with one double root $\lambda_2 = \lambda_3$ yields failure for $0 \leq \lambda < \lambda_1$ and is safe otherwise. At $\lambda = \lambda_2 = \lambda_3$, the safety margin is null. Again, the global optimum is λ_1, and $\lambda_2 = \lambda_3$ appears as local optimum.

The largest positive root of the used scaling criterion corresponds to the inferior boundary of the last interval, which extends theoretically to ∞. This solution, which may be local or global, is commonly found in optimization software [1, 7, 8]. Our study showed that the global optimum is equal to λ_1, the lower boundary of the first safe design interval, which may be a closed one. Therefore, assuming that any scaling factor $\lambda \geq \lambda_1$ leads to a safe design is wrong! For instance, doubling the thickness of a safe laminate may lead to catastrophic failure instead of added safety, if this doubled thickness falls within the second failure interval!

These conclusions are based on strain paths and the convex nature of the used failure criterion; they are hence quite general and also applicable with other convex criteria.

4.3 Safe design intervals with integer layups

Instead of resorting to integer programming which is very complex and tricky, our comprehensive scaling theory deals with real numbers and can then post-process the results, by rounding-off appropriately, while not approximating!

Introducing $\lfloor x \rfloor$ as the floor or rounded-down integer value of x, and $\lceil x \rceil$ as the ceil or rounded-up integer value of x, the integer safe design intervals for λ are easily obtained.

In the case of one interval, the safe design allowable range is

$$[\lceil \lambda_1 \rceil; \infty[\tag{18}$$

In the case of two intervals, the safe design allowable ranges are

$$[\lceil \lambda_1 \rceil; \lfloor \lambda_2 \rfloor] \quad \text{and} \quad [\lceil \lambda_3 \rceil; \infty[\tag{19}$$

Caution is necessary at this stage! If $\lceil\lambda_1\rceil > \lfloor\lambda_2\rfloor$, then the first interval vanishes altogether! So, we can face a situation with two real intervals, but with only one integer interval left over (Example in Section 5).

In practice, the rounding-off procedure is more sophisticated. It adjusts λ minimally, just so that all numbers of plies in the layup are integer. For instance, with a $[0_{4\lambda}/90_{4\lambda}]$ or $[50_{8\lambda}/60_{12\lambda}]$ layup, .00, .25, .50, and .75 are all acceptable fractional parts. However, with $[0_{4\lambda}/90_{3\lambda}]$, only .00 is acceptable.

5 Application

ESACLOSE (ESA Composite Laminates and Sandwiches Optimization Scaling Engine) [4] computes the strength design allowables, in real and integer numbers of plies.

Let us further use our $[0_{4\lambda}/90_{4\lambda}]$ laminate made of $T300/N5208$ with four loadcases. The loadcases and the corresponding safe scaling intervals are

```
          [MN/m]                [MN m/m]
     N1      N2      N6      M1      M2      M6  SAFE SCALING INTERVALS (REAL LAYUP)
---------------------  -------------------  ------------------------------------
  2.000   0.000   0.000 -0.003   0.000   0.000  [  4.79 -   7.24] + [ 23.96 - inf[
  2.000   0.000   0.000 -0.004   0.000   0.000  [  5.70 - 12.13] + [ 19.07 - inf[
  2.000   0.000   0.000 -0.005   0.000   0.000  [  6.50 -    inf[
  2.000   0.000   0.000 -0.006   0.000   0.000  [  7.22 -    inf[

All loadcases (Global) :                        [  7.22 -   7.24] + [ 23.96 - inf[
==========================================  ====================================

     N1      N2      N6      M1      M2      M6  SAFE SCALING INTERVALS (INT. LAYUP)
---------------------  -------------------  ------------------------------------
  2.000   0.000   0.000 -0.003   0.000   0.000  [  5.00 -   7.00] + [ 24.00 - inf[
  2.000   0.000   0.000 -0.004   0.000   0.000  [  5.75 - 12.00] + [ 19.25 - inf[
  2.000   0.000   0.000 -0.005   0.000   0.000  [  6.50 -    inf[
  2.000   0.000   0.000 -0.006   0.000   0.000  [  7.25 -    inf[

All loadcases (Global - int. layup):                                [ 24.00 - inf[
==========================================  ====================================
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This example shows that $\lambda_{optimum} = 7.22$ in real numbers. For the integer layup, it becomes $\lambda_{optimum} = 24.00$ because $(\lceil 7.22\rceil = 7.25) \not\leq (7.00 = \lfloor 7.24\rfloor)$.

Considering only the first two loadcases, the optima become 5.70 and 5.75 respectively.

6 Conclusions

We showed how the mechanisms of strain distribution in a laminate are affected by proportional thickness scaling, which can make the neutral plane shift across layers.

Having one or two safe design zones implies major consequences, like the potential existence of local optima. More, taking any thickness scale larger than the global optimum is just not safe. If it falls in between the two safe zones, it means catastrophic failure! Manufacturing tolerance is hence also affected. Depending on where the nominal design lies in the safe design intervals, admissible tolerances may have to be restrained. Besides, the first thickness allowable range may be very narrow. Consequently, applying a factor of safety to the nominal design may actually worsen rather than improve its safety margin.

All this follows from the combination of in-plane and bending load induced strains, that is ε_i^N and ε_i^M, of opposite signs, as demonstrated by the sign and trend analysis of scaled strains. A few remarkable scales were pin-pointed, such as the zeroing, critical, and inflexion thicknesses/strains, which appear in the exact ratio of 1, 2, 3.

On the other hand, adopting the highest root, that is the potentially local optimum, selects the lightest design which can always be safely thickened. Such choice is however made at the expense of weight: in our example, the local optimum is five times heavier than the global one.

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