THEORY OF ELLIPTICAL SANDWICH CYLINDRICAL SHELLS WITH DISSIMILAR FACINGS

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SUMMARY: The Sanders-type theory of elliptical sandwich shells with dissimilar facings is formulated. The governing equations account for transverse shear strains and for rotations about the normal to the middle surface of the shell. The constitutive equations correspond to a sandwich shell where each facing is formed of regular symmetrically laminated layers. Accordingly, the matrix of extensional, coupling and bending stiffnesses is fully populated, except for the elements A_{16} and A_{26} that are equal to zero. The governing equations for the elliptical sandwich shell are reduced to the corresponding results for a circular cylindrical shell if the radius of curvature of the shell is constant.

KEYWORDS: sandwich structures, elliptical shells, Sanders' shell theory.

INTRODUCTION

Elliptical and other noncircular cylindrical shells can offer an economical and rational solution for fuselages of cargo planes as well as in other applications. Mathematical difficulties in treatment of the problem are the primary reason why research in this area has been scant. Isotropic oval shells were considered by Vafakos et al [1,2], including also reinforced configurations [3,4], and by Kempner and Chen [5,6]. Isotropic elliptical cylindrical shells have also been considered [7]. Daugherty and Vinson presented an analytical solution for transversely isotropic shear deformable noncircular cylinders described by the Love-Reissner type theory of shells [8]. Governing equations that can be applied to the analysis of composite elliptical shells have been presented in the book of Grigorenko and Vasilenko [9].

The recent papers of Meyers and Hyer presented analytical and experimental studies of clamped elliptical composite cylinders subjected to internal pressure or axial compression [10,11]. The analytical model utilizing the energy method was developed by assuming that the shell is thin. The curvature of the elliptical cross section was approximated by a cosine series.

To the best knowledge of the authors, the studies of shear-deformable elliptical sandwich shells have not been published. The theory developed in this paper is applicable to composite elliptical sandwich cylinders with dissimilar facings subject to mechanical loading. Transverse shearing deformations in the core and in the facings are accounted for. The formulation is based on the Sanders shell theory [12]. The paper represents a generalization of the earlier work of the authors on circular cylindrical sandwich shells with dissimilar facings [13].

GOVERNING EQUATIONS FOR A SANDWICH ELLIPTICAL SHELL

The following formulation is based on the improved first-approximation Sanders' shell theory [12]. The strain-displacement relations for an elliptical shell are given, according to this theory, as

$$
\begin{aligned}
\varepsilon_{11}^{0} &= u_{,x} + \frac{1}{2}\phi_1^2 + \underline{\frac{1}{2}\phi^2} \\
\varepsilon_{22}^{0} &= v_{,s} + \frac{w}{R} + \frac{1}{2}\phi_2^2 + \underline{\frac{1}{2}\phi^2} \\
\gamma_{12}^{0} &= v_{,x} + u_{,s} + \phi_1\phi_2 \\
\kappa_{11} &= \phi_{1,x} \\
\kappa_{22} &= \phi_{2,s} \\
2\kappa_{12} &= \phi_{2,x} + \phi_{1,s} + \underline{\frac{1}{R}\phi} \\
\gamma_{13} &= \phi_1 + w_{,x} \\
\gamma_{23} &= \phi_2 + w_{,s} - \frac{v}{R}
\end{aligned}
\tag{1}
$$

where ε_{ij}^{0} are the middle surface strains, κ_{ij} are the changes of curvature and twist of the middle surface, γ_{i3} are transverse shear strains, u, v and w denote in-surface displacements in the axial and tangential directions (u and v) and the transverse deflection in the direction perpendicular to the surface. The middle surface radius R = R(s) is a function of the tangential coordinate, ϕ_1 and ϕ_2 are the rotations about the in-surface coordinate axes and ϕ is the rotation about the normal to the surface that is taken here in the form

$$\phi = \frac{1}{2}(v_{,x} - u_{,s}) \tag{2}$$

Note that the terms underlined in equations (1) are retained only, if the rotations about the normal to the middle surface of the shell are included into the analysis. The total strains in the shell at a distance z from the middle surface are given by

$$\varepsilon_{11} = \varepsilon_{11}^{0} + z\,\kappa_{11}$$

$$\varepsilon_{22} = \varepsilon_{22}^{0} + z\,\kappa_{22}$$

$$\gamma_{12} = \gamma_{12}^{0} + 2z\,\kappa_{12} \tag{3}$$

The equations of equilibrium of an elliptical shell can be obtained from the general shell equations of Sanders [12] in the form

$$\begin{aligned}
&N_{11,x} + N_{12,s} - \underline{\frac{1}{2R^2}(M_{12,s}R - M_{12}R_{,s})} + p_1 = 0 \\
&N_{12,x} + N_{22,s} + \frac{1}{R}(Q_{23} + \frac{1}{2}M_{12,x}) + p_2 = 0 \\
&Q_{13,x} + Q_{23,s} - \frac{N_{22}}{R} + p = 0 \\
&M_{11,x} + M_{12,s} - Q_{13} = 0 \\
&M_{12,x} + M_{22,s} - Q_{23} = 0
\end{aligned} \tag{4}$$

where N_{ij}, Q_{ii}, and M_{ij} denote in-surface and transverse shear stress resultants and stress couples, respectively, the underlined terms account for the rotations about the normal to the middle surface and p_1, p_2, p are the loading terms that can also include the inertial contributions.

The solution of equations of equilibrium must satisfy the boundary conditions along the curved edges x = const. These conditions are listed here for the edges supported by the stiffeners that are rigid in the web plane (x = const) but flexible in the direction perpendicular to the web. Blade stiffeners can serve as an example of such stiffeners where

$$\begin{aligned}
&N_{11} = \overline{N}_{11}^{*} \\
&N_{12} + \frac{3}{2R}M_{12} = \overline{N}_{12}^{*} \\
&w = \phi_2 = M_{11} = 0
\end{aligned} \tag{5}$$

The quantities with the overbar and the asterisk that appear in equations (5) represent applied loads.

If the elliptical shell is not closed in the tangential direction, i.e. the elliptical shell section is supported by stiffeners along the straight edges s = const, the corresponding boundary conditions may be formulated in the form

$$v = u = \phi_1 = w = M_{22} = 0 \tag{6}$$

Note that conditions (6) reflect the same type of stiffeners as those along the curved edges x = const. However, the displacement in the tangential direction perpendicular to the plane of the stiffener is taken equal to zero to reflect the restraint introduced by the adjacent shell structure.

The constitutive relations employed in this paper correspond to a sandwich shell with dissimilar facings, each of them formed from regular symmetrically laminated layers. Accordingly only the extensional stiffness matrix elements $A_{16} = A_{26} = 0$, and the constitutive relations become

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & \text{sym.} & & D_{12} & D_{22} & D_{26} \\ & & & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{0} \\ \varepsilon_{22}^{0} \\ \gamma_{12}^{0} \\ \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix} \qquad (7)$$

$$Q_{13} = k\, A_{55}$$

$$Q_{23} = k\, A_{44}$$

where the stiffness elements are given by

$$\left\{A_{ij}, B_{ij}, D_{ij}\right\} = \int_{h_{f1}+h_{f2}+h_c} Q_{ij} \left\{1, z, z^2\right\} dz \qquad i,j = 1, 2, 6$$

$$A_{rr} = k \int_{h_c} Q_{rr}\, dz \qquad r = 4, 5 \qquad (8)$$

In equations (8), Q_{ij} are transverse reduced stiffnesses, k is the shear correction factor, and the integration is carried out over the thickness of the facings (h_{f1}, h_{f2}) and over the core thickness (h_c). Note that although all elements of the sandwich shell introduced above can resist transverse shear forces, the core is not capable of resisting in-surface stresses. This is rather typical for the case where the core is manufactured from honeycomb or foam.

The shear correction factors k have been considered by a number of researchers. For example, Reissner [14] suggested to use k = 5/6, while Mindlin [15] obtained $k = \pi^2/12$. Gordaninejad and Bert [16] obtained the shear correction factor for thick sandwich beams from the equilibrium equations of elasticity and the concept of equivalent strain energy. The factor was equal to 0.238 for beams with glass/epoxy facings and to 0.382 for beams with slightly bimodular graphite/epoxy facings. In both cases, aluminum-foil honeycomb cores were considered. In the case of bimodular aramid cord-rubber facings and a bimodular polyurethane foam core, the shear correction factor was equal to 0.804. As was shown in [16], the classical sandwich theory operating with the value of the shear correction factor equal to 1.0 is unconservative. The study of Greenberg and Stavsky [17] resulted in the conclusion that small variations of the shear correction factors do not noticeably affect the results obtained for cylindrical shells.

The substitution of equations (7) into the equations of equilibrium (4) results in a set of linear differential equations with respect to the elements of the vector of displacements and rotations:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \phi_1 \\ \phi_2 \end{Bmatrix} + \begin{Bmatrix} p_1 \\ p_2 \\ p \\ 0 \\ 0 \end{Bmatrix} = 0 \qquad (9)$$

The linear differential operators L_{ij} are given in the Appendix. As is typical for the Sanders' shell theory, the matrix of these operators is not symmetric.

The particular case of a circular cylindrical shell is obtained if the radius of the middle surface R = const. It is easy to see that the corresponding simplifications of both the operators L_{ij} and the constitutive relations result in the formulation presented for cylindrical shells in the earlier paper by Birman and Simitses [13].

CONCLUSIONS

The paper presents the theory of elliptical sandwich shells with dissimilar facings. The formulation is based on the improved first-approximation Sander's shell theory and accounts for transverse shear strains and for rotations about the normal to the middle surface of the shell. The constitutive relationships used in the paper correspond to the practical case where the layers of each facing are symmetrically laminated with respect to the middle surface of this facing. Accordingly, only extensional stiffnesses A_{16} and A_{26} are equal to zero. The system of linear differential equations in terms of in-surface and transverse displacements and rotations about in-surface coordinate axes is reduced to the corresponding system for a circular cylindrical sandwich shell if the shell radius is constant.

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REFERENCES

1. Vafakos, W.P., Romano, F. and Kempner, J., "Clamped Short Oval Cylindrical Shells Under Hydrostatic Pressure," *Journal of the Aerospace Sciences*, Vol. 29, 1962, pp. 1347-1357.

2. Vafakos, W.P., Nissel, N. and Kempner, J., "Energy Solutions for Simply-Supported Oval Shells," *AIAA Journal*, Vol. 2, 1964, pp. 555-557.

3. Vafakos, W.P., Nissel, N. and Kempner, J., "Pressurized Oval Cylinders with Closely Spaced Rings," *AIAA Journal*, Vol. 4, 1966, pp. 338-345.

4. Nissel, N. and Vafakos, W.P., “Assymptotic Solution for Short Ring – Reinforced Oval Cylinders,” *AIAA Journal*, Vol. 6, 1968, pp. 133-140.

5. Kempner, J. and Chen, Y.-N., “Large Deflections of an Axially Compressed Oval Cylindrical Shell,” *Proceedings of the XI International Congress of Applied Mechanics*, Munich, Germany, (1964), Springer-Verlag, Berlin, 1965, pp. 299-305.

6. Kempner, J. and Chen, Y.-N., “Buckling and Postbuckling of an Axially Compressed Oval Cylindrical Shell,” *Proceedings of the Symposium on the Theory of Shells (L.H. Donnell 70th Anniversary Volume)*, Houston, 1967, pp. 139-183.

7. Sewall, J.L. and Pusey. C.G., “Vibration Study of Clamped-Free Elliptical Cylindrical Shells,” *AIAA Journal*, Vol. 9, 1971, pp. 1004-1011.

8. Daugherty, R.L. and Vinson, J.R., “Asymptotic Solution for Noncircular Cylinders of Pyrolitic Graphite,” Developments in Mechanics, Vol. 5, Proceedings of the 11th Midwestern Conference, 1969, pp. 563-576.

9. Grigorenko, Ya.M. and Vasilenko, A.T. *Problems of Statics of Anisotropic Heterogeneous Shells*, Nauka, Moscow, 1992. In Russian.

10. Meyers, C.A. and Hyer, M.W., “Response of Elliptical Composite Cylinders to Internal Pressure Loading,” *Mechanics of Composite Materials and Structures*, Vol. 4, 1997, pp. 317-343.

11. Meyers, C.A. and Hyer, M.W., “Response of Elliptical Composite Cylinders to Axial Compression Loading,” *Mechanics of Composite Materials and Structures*. In press.

12. Sanders, J.L., Jr., *An Improved First Approximation Theory for Thin Shells*, NACA Report 24, Washington, D.C., 1959.

13. Birman, V. and Simitses, G.J., “Theory of Box-Type Sandwich Shells with Dissimilar Facings Subjected to Thermomechanical Loads,” submitted for publication to *AIAA Journal*. The conference version of the same paper was published in *Recent Advances in Mechanics of Aerospace Structures and Materials*, Ed. Sankar, B.V., ASME (American Society of Mechanical Engineers), New York, 1998, pp. 151-163.

14. Reissner, E., “The Effect of Transverse Shear Deformation on the Bending of Elastic Plate,” *ASME Journal of Applied Mechanics*, Vol. 12, 1945, pp. A69-A77.

15. Mindlin, R.D., “Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates,” *ASME Journal of Applied Mechanics*, Vol. 18, 1951, pp. 31-38.

16. Gordaninejad, F. and Bert, C.W., “A New Theory for Bending of Thick Sandwich Beams,” *International Journal of Mechanical Sciences*, Vol. 31, 1989, pp. 925-934.

17. Greenberg, J.B. and Stavsky, Y., "Vibrations of Axially Compressed Laminated Orthotropic Cylindrical Shells, Including Shear Deformation," *Acta Mechanica*, Vol. 37, 1980, pp. 13-28.

18. Simitses, G.J., Shaw, D. and Sheinman, I., "Stability of Cylindrical Shells by Various Nonlinear Theories," *ZAMM*, 1985, Vol. 65, pp. 159-166.

APPENDIX

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} - \frac{B_{16}}{R}\frac{\partial^2}{\partial x \partial s} + (A_{66} - \frac{B_{66}}{R} + \frac{D_{66}}{4R^2})\frac{\partial^2}{\partial s^2}$$
$$+ \frac{R_{,s}}{2R^2}[B_{16}\frac{\partial}{\partial x} + (2B_{66} - \frac{D_{66}}{R})\frac{\partial}{\partial s}]$$

$$L_{12} = \frac{B_{16}}{2R}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66} - \frac{D_{66}}{4R^2})\frac{\partial^2}{\partial x \partial s} - \frac{B_{26}}{2R}\frac{\partial^2}{\partial s^2}$$
$$+ \frac{R_{,s}}{2R^2}[(-B_{66} + \frac{D_{66}}{R})\frac{\partial}{\partial x} + B_{26}\frac{\partial}{\partial s}]$$

$$L_{13} = \frac{A_{12}}{R}\frac{\partial}{\partial x} - \frac{B_{26}}{2R^2}\frac{\partial}{\partial s} + \frac{R_{,s}}{R^3}B_{26}$$

$$L_{14} = B_{11}\frac{\partial^2}{\partial x^2} + (2B_{16} - \frac{D_{16}}{2R})\frac{\partial^2}{\partial x \partial s} + (B_{66} - \frac{D_{66}}{2R})\frac{\partial^2}{\partial s^2}$$
$$+ \frac{R_{,s}}{2R^2}(D_{16}\frac{\partial}{\partial x} + D_{66}\frac{\partial}{\partial s})$$

$$L_{15} = B_{16}\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66} - \frac{D_{66}}{2R})\frac{\partial^2}{\partial x \partial s} + (B_{26} - \frac{D_{26}}{2R})\frac{\partial^2}{\partial s^2}$$
$$+ \frac{R_{,s}}{2R^2}(D_{66}\frac{\partial}{\partial x} + D_{26}\frac{\partial}{\partial s})$$

$$L_{21} = \frac{B_{16}}{2R}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66} - \frac{D_{66}}{4R^2})\frac{\partial^2}{\partial x \partial s} - \frac{B_{26}}{2R}\frac{\partial^2}{\partial s^2}$$
$$+ \frac{R_{,s}}{2R^2}B_{26}\frac{\partial}{\partial s}$$

$$L_{22} = (A_{66} + \frac{B_{66}}{R} + \frac{D_{66}}{4R^2})\frac{\partial^2}{\partial x^2} + \frac{B_{26}}{R}\frac{\partial^2}{\partial x \partial s} + A_{22}\frac{\partial^2}{\partial s^2}$$
$$- \frac{kA_{44}}{R^2} - \frac{R_{,s}}{2R^2}B_{26}\frac{\partial}{\partial x}$$

$$L_{23} = \frac{B_{26}}{2R^2}\frac{\partial}{\partial x} + \frac{1}{R}(A_{22} + kA_{44})\frac{\partial}{\partial s} - \frac{R_{,s}}{R^2}A_{22}$$

$$L_{24} = (B_{16} + \frac{D_{16}}{2R})\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66} + \frac{D_{66}}{2R})\frac{\partial^2}{\partial x \partial s} + B_{26}\frac{\partial^2}{\partial s^2}$$

$$L_{25} = (B_{66} + \frac{D_{66}}{2R})\frac{\partial^2}{\partial x^2} + (2B_{26} + \frac{D_{26}}{2R})\frac{\partial^2}{\partial x \partial s} + B_{22}\frac{\partial^2}{\partial s^2} + \frac{kA_{44}}{R}$$

$$L_{31} = -\frac{A_{12}}{R}\frac{\partial}{\partial x} + \frac{B_{26}}{2R^2}\frac{\partial}{\partial s}$$

$$L_{32} = -\frac{B_{26}}{2R^2}\frac{\partial}{\partial x} - \frac{1}{R}(A_{22} + kA_{44})\frac{\partial}{\partial s} + \frac{R_{,s}}{R^2}kA_{44}$$

$$L_{33} = kA_{55}\frac{\partial^2}{\partial x^2} + kA_{44}\frac{\partial^2}{\partial s^2} - \frac{A_{22}}{R^2}$$

$$L_{34} = (kA_{55} - \frac{B_{12}}{R})\frac{\partial}{\partial x} - \frac{B_{26}}{R}\frac{\partial}{\partial s}$$

$$L_{35} = -\frac{B_{26}}{R}\frac{\partial}{\partial x} + (kA_{44} - \frac{B_{22}}{R})\frac{\partial}{\partial s}$$

$$L_{41} = B_{11}\frac{\partial^2}{\partial x^2} + (2B_{16} - \frac{D_{16}}{2R})\frac{\partial^2}{\partial x \partial s} + (B_{66} - \frac{D_{66}}{2R})\frac{\partial^2}{\partial s^2}$$
$$+ \frac{R_{,s}}{2R^2}D_{66}\frac{\partial}{\partial s}$$

$$L_{42} = (B_{16} + \frac{D_{16}}{2R})\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66} + \frac{D_{66}}{2R})\frac{\partial^2}{\partial x \partial s} + B_{26}\frac{\partial^2}{\partial s^2}$$
$$- \frac{R_{,s}}{2R^2}D_{66}\frac{\partial}{\partial x}$$

$$L_{43} = (\frac{B_{12}}{R} - kA_{55})\frac{\partial}{\partial x} + \frac{B_{26}}{R}\frac{\partial}{\partial s} - \frac{R_{,s}}{R^2}B_{26}$$

$$L_{44} = D_{11}\frac{\partial^2}{\partial x^2} + 2D_{16}\frac{\partial^2}{\partial x \partial s} + D_{66}\frac{\partial^2}{\partial s^2} - kA_{55}$$

$$L_{45} = D_{16}\frac{\partial^2}{\partial x^2} + (D_{12} + D_{66})\frac{\partial^2}{\partial x \partial s} + D_{26}\frac{\partial^2}{\partial s^2}$$

$$L_{51} = B_{16}\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66} - \frac{D_{66}}{2R})\frac{\partial^2}{\partial x \partial s} + (B_{26} - \frac{D_{26}}{2R})\frac{\partial^2}{\partial s^2} + \frac{R_{,s}}{2R^2}D_{26}\frac{\partial}{\partial s}$$

$$L_{52} = (B_{66} + \frac{D_{66}}{2R})\frac{\partial^2}{\partial x^2} + (2B_{26} + \frac{D_{26}}{2R})\frac{\partial^2}{\partial x \partial s} + B_{22}\frac{\partial^2}{\partial s^2} + \frac{kA_{44}}{R} - \frac{R_{,s}}{2R^2}D_{26}\frac{\partial}{\partial x}$$

$$L_{53} = \frac{B_{26}}{R}\frac{\partial}{\partial x} + (\frac{B_{22}}{R} - kA_{44})\frac{\partial}{\partial s} - \frac{R_{,s}}{R^2}B_{22}$$

$$L_{54} = L_{45}$$

$$L_{55} = D_{66}\frac{\partial^2}{\partial x^2} + 2D_{26}\frac{\partial^2}{\partial x \partial s} + D_{22}\frac{\partial^2}{\partial s^2} - kA_{44}$$

where

$$R_{,s} = \frac{\partial R}{\partial s}$$