INTERACTION BETWEEN A HOLE AND AN ELASTIC CIRCULAR INCLUSION UNDER A REMOTE UNIFORM HEAT FLOW

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SUMMARY: The plane problem for a circular elastic inclusion interacted with an arbitrarily oriented hole under a remote uniform heat flow, is solved in this work. The proposed method is based on the complex variable theory and the existing solutions for dislocation functions which permit us to formulate boundary integral equations with a weak singular kernel. The unknown coefficients leaving in the singular integral equations can thus be solved numerically by applying the appropriate interpolation formulae. The hoop stresses, which are directly related to the coefficients of dislocation functions, are then determined accordingly. Several numerical examples are given to demonstrate the use of the present approach. Comparisons between the calculated numerical results and the corresponding existing solutions show that the method proposed here is effective, simple and general.

KEYWORDS: circular elastic inclusion, dislocation function.

INTRODUCTION

Studies on the interaction among multiple inclusions or inhomogeneities have been received much attention in research communities. Exact closed form solutions can only be found for a single inclusion embedded in an infinite isotropic or anisotropic medium [1-3] whereas no exact closed form solutions are available to the corresponding problem of multiple inclusions with two or more separate interfaces. Very few solutions of the corresponding thermoelastic problem appear in the literature. Recently, Chao et al. [4] derived the general expressions of the complex potentials for the thermoelastic problem with multiple inclusions which satisfy the prescribed continuity conditions for each circular inclusion. The remaining unknown coefficients appearing in a system of coupled algebraic equations were solved by the perturbation technique. In this paper a unified approach is proposed to solve the problem with a hole of any arbitrary shape interacted with an elastic circular inclusion under remote uniform heat flow. The hole contour is simulated as a polygon of N line segments using the appropriate interpolation formulas. The hoop stress along the hole boundary can be determined in terms of the unknown coefficients appearing in the interpolation formulas. Numerical examples associated with a circular hole interacting with the circular inclusion under a remote uniform heat flow are examined in detail. The obtained results in this study will be helpful in deep understanding the thermoelastic interaction behavior when an existing defect such as holes and the surrounding inclusions become close to each other.

THE TEMPERATURE FIELD

Consider two homogeneous, isotropic elastic materials. Let one occupy the infinite region S_1, exterior to the circle of radius a, whereas the other occupies the region S_2, interior to the circle of radius a. The governing equations for two-dimensional steady state heat conduction problems are given by

$$\nabla^2 T_j(x, y) = 0, \;\; j = 1, 2$$

with boundary conditions $Q_1 = Q_2$ and $T_1 = T_2$ along the interface $|z| = a$ where the resultant heat flux $Q_j\,(j = 1,2)$ and temperature $T_j\,(j = 1,2)$ for each medium are expressed in terms of the complex potential $g_j'(z)$ as

$$Q_j = \int (q_{xj}dy - q_{yj}dx) = -k_j \,\mathrm{Im}[g_j'(z)] \tag{1}$$

$$T_j = \mathrm{Re}[g_j'(z)] \tag{2}$$

where Re and Im stand for the real part and imaginary part of the bracketed expression, respectively. The quantities q_{xj} and q_{yj} represent the components of the heat flux in the $x-$ and $y-$directions, respectively, and k_j denotes the heat conductivities. We consider now a circular elastic inclusion perfectly bonded to an infinite matrix subjected to a thermal field whose sources are in the matrix (including infinity) so that the thermal field is free of singularities inside or on the boundary of the circular inclusion. The proposed solution can be expressed as

$$g_1'(z) = g_0'(z) + \underset{\sim}{g_1'}(z) \tag{3}$$

$$g_2'(z) = \underset{\sim}{g_2'}(z) \tag{4}$$

where $g_0'(z)$ stands for the temperature function associated with the unperturbed field while $\underset{\sim}{g_1'}(z)$ (or $\underset{\sim}{g_2'}(z)$) is the temperature function associated with the perturbed field of matrix (or inclusion). Based on the property of analytical continuation, we have the following results

$$g_1'(z) = g_0'(z) + \frac{k_1 - k_2}{k_1 + k_2}\overline{g_0'}(a^2 / z) \tag{5}$$

$$g_2'(z) = \frac{2k_1}{k_1 + k_2} g_0'(z) \tag{6}$$

THE STRESS FIELD

We consider a circular inclusion perfectly bonded to an infinite matrix subjected to an elastic field whose sources are in the matrix. The proposed solution can be expressed as

$$\phi_1(z) = \phi_0(z) + \underset{\sim}{\phi_1}(z), \quad \psi_1(z) = \psi_0(z) + \underset{\sim}{\psi_1}(z) \tag{7}$$

$$\phi_2(z) = \underset{\sim}{\phi_2}(z), \quad \psi_2(z) = \underset{\sim}{\psi_2}(z) \tag{8}$$

where $\phi_0(z)$ and $\psi_0(z)$ represent the stress functions associated with the unperturbed field while $\underset{\sim}{\phi_1}(z), \underset{\sim}{\psi_1}(z)$ (or $\underset{\sim}{\phi_2}(z), \underset{\sim}{\psi_2}(z)$) are the functions corresponding to the perturbed field of matrix (or inclusion). Similar to the previous derivations, we finally have the results as follows

$$\phi_1(z) = \phi_0(z) + \gamma_3[z\overline{\phi_0'}(a^2/z) - z\overline{\phi_0'(0)} + \overline{\psi_0}(a^2/z)] + \gamma_2\,\overline{f_0}(a^2/z) \tag{9}$$

$$\psi_1(z) = \psi_0(z) + \gamma_1\overline{\phi_0}(a^2/z) + \gamma_3\frac{a^4}{z^3}[\overline{\psi_0'}(a^2/z) + z\overline{\phi_0''}(a^2/z) - \frac{z^2}{a^2}\overline{\phi_0'}(a^2/z)]$$

$$+\gamma_4\overline{g_0}(a^2/z) + \frac{a^4}{z^3}\gamma_2\,\overline{f_0'}(a^2/z) + \frac{a^2}{z}\{[\frac{1+\gamma_1}{1-\gamma_3^*}\gamma_3^* + \gamma_3]\overline{\phi_0'(0)}$$

$$+[\frac{1+\gamma_1}{1-\gamma_3^*} - 1]\phi_0'(0) + \frac{\gamma_4}{1-\gamma_3^*}[\gamma_3^*\overline{g_0'(0)} + g_0'(0)]\} \tag{10}$$

$$\phi_2(z) = (1+\gamma_1)\phi_0(z) + \gamma_4 g_0(z) + \frac{\gamma_3^*}{1-\gamma_3^{*2}}z\{(1+\gamma_1)[\gamma_3^*\phi_0'(0) + \overline{\phi_0'(0)}]$$

$$+\gamma_4[\gamma_3^* g_0'(0) + \overline{g_0'(0)}]\} \tag{11}$$

$$\psi_2(z) = (1+\gamma_3)[\frac{a^2}{z}\phi_0'(z) - \frac{a^2}{z}\phi_0'(0) + \psi_0(z)] - \frac{a^2}{z}\phi_2'(z) + \gamma_2 f_0(z)$$

$$+\frac{a^2}{z}\phi_2'(0) \tag{12}$$

where

$$\gamma_1 = \frac{\kappa_1\mu_2 - \kappa_2\mu_1}{\kappa_2\mu_1 + \mu_2}, \quad \gamma_2 = \frac{2\mu_1\mu_2\beta_1}{\kappa_1\mu_2 + \mu_1}\frac{k_2 - k_1}{k_1 + k_2}, \quad \gamma_3 = \frac{\mu_2 - \mu_1}{\kappa_1\mu_2 + \mu_1}$$

$$\gamma_4 = \frac{2\mu_1\mu_2}{\kappa_2\mu_1 + \mu_2}(\beta_1 - \frac{2k_1}{k_1 + k_2}\beta_2), \quad \gamma_3^* = \frac{\mu_1 - \mu_2}{\kappa_2\mu_1 + \mu_2} \tag{13}$$

FORMULATION OF INTEGRAL EQUATIONS FOR AN INSULATED HOLE

In this section we consider a single hole L, which is embedded in a matrix, interacting with an elastic circular inclusion. The corresponding homogeneous solutions associated with a single hole can be represented by distributed temperature dislocations and edge dislocations along the hole border as

$$g_0'(z) = -\frac{i}{2\pi}\int_L b_0(s)\log(z-t)ds \tag{14}$$

$$\phi_0(z) = \frac{i\mu_1}{\pi(1+\kappa_1)}\int_L [b_1(s)+ib_2(s)]\log(z-t)ds \tag{15}$$

$$\psi_0(z) = \frac{-i\mu_1}{\pi(1+\kappa_1)}\int_L [b_1(s)-ib_2(s)]\log(z-t)ds - \frac{i\mu_1}{\pi(1+\kappa_1)}\int_L \frac{[b_1(s)+ib_2(s)]\bar{t}}{z-t}ds \tag{16}$$

where $b_0(s)$ stands for the strength of the temperature dislocation and $b_1(s), b_2(s)$ represent the components of the displacement discontinuities across the dislocation line. The temperature function $b_0(s)$ can be found from the thermal boundary condition of an insulated hole such that the total heat flux across the hole surface must be balanced by the given resultant heat flux Q_1 across the hole border L in the unflawed media, i.e.

$$Q_1 = -k_1 \operatorname{Im}[g_1'(t)] + c_0, t \in L \tag{17}$$

where c_0 is a constant. In addition, the single-valued condition of the temperature must be satisfied, i.e.

$$\int_L b_0(s)ds = 0 \tag{18}$$

Substitution of Eq. (14) into Eq. (5) and applying Eqs. (17) and (18) results in the singular integral equation for solving the unknown function $b_0(s)$. On the other hand, the unknown functions $b_1(s), b_2(s)$ can be determined from the traction-free boundary condition such that the force acting on the hole surface must be balanced by the given resultant force applied on the hole border, i.e.

$$-Y_1 + iX_1 = \phi_1(t) + t\overline{\phi_1'(t)} + \overline{\psi_1(t)} + c_1 + ic_2, t \in L \tag{19}$$

where c_1 and c_2 are real constants to be determined. Moreover, the requirement of single-valued displacements given by

$$\int_L [b_1(s) + ib_2(s)]ds = \int_L \beta_1 [\int b_0(\xi)d\xi]ds \tag{20}$$

must be satisfied. Substitution of Eqs. (15) and (16) into Eqs. (9) and (10), and applying Eqs. (19) and (20) yields the singular integral equation for solving the unknown functions $b_1(s), b_2(s)$.

NUMERICAL RESULTS AND DISCUSSIONS

The dislocation functions $b_0(s), b_1(s)$ and $b_2(s)$ appearing in the above singular integral equations will be solved numerically using the appropriate interpolation formulas. For the purpose of performing the numerical calculation, the contour L is replaced by a polygon of N line segments. The interpolation formulas for line segments in local coordinates $s_j (1 \le j \le N)$ are taken as

$$b_i(s_j) = b_{i,j} \frac{d_j - s_j}{2d_j} + b_{i,j+1} \frac{d_j + s_j}{2d_j} \quad (i = 0,1,2) \tag{21}$$

where $d_j (1 \le j \le N)$ are the half-length for each line segment and $b_{i,j} (0 \le j \le N)$ are the unknown coefficients to be determined. If the preceding formulas are used, the boundary integral equation for the temperature function together with the subsidiary condition can be carried out to yield $N+2$ algebraic equations for solving $N+2$ unknown coefficients $(b_{0,0}, b_{0,1}, b_{0,2}, ..., b_{0,N}, c_0)$. Similarly, the boundary integral equation for the stress functions together with the subsidiary condition can be arranged to yield $2N+4$ algebraic equations for solving $2N+4$ unknown coefficients $(b_{1,0}, b_{1,1}, ..., b_{1,N}, b_{2,0}, b_{2,1}, ...b_{2,N}, c_1, c_2)$. Once the stress functions are obtained, the hoop stresses along the hole boundary can be evaluated by

$$\sigma_{\theta\theta} = 4\,\mathrm{Re}[\phi'(z)], \tag{22}$$

We consider an insulated circular hole interacting with an elastic circular inclusion under a remote uniform heat flow (Fig. 1). In order to perform the numerical technique, the contour of the circular hole is replaced by a polygon of N line elements discreted with a number of N points expressed by

$$x_i = a\cos[\frac{2(i-1)\pi}{N}], \;\; y_i = a\sin[\frac{2(i-1)\pi}{N}], \quad (x_i, y_i) \in L \tag{23}$$

The calculated hoop stresses along the circular hole boundary with the number of line segments $N = 40$, which are checked to achieve a good accuracy with an error less than 1 percent as compared to those obtained by Chao et al. [5] with $d/a = 4$, are displayed in Figs. 2 – 3. It is interesting to see that, as a hole approaches the circular inclusion with the distance $d/a = 2.2$ for the case $\phi = 45^0$, the maximum absolute hoop stress takes place at the point $\theta = 200^0$, which is not exactly the point $\theta = 225^0$ nearest to the neighboring inclusion. It is expected that, as a hole is infinitely close to the neighboring circular inclusion, for example $d/a = 2.11$ as displayed in Fig. 3, the maximum hoop stress would occur at the point along a hole boundary which is closest to the neighboring inclusion.

CONCLUDING REMARKS

In this article the interaction between an arbitrarily located hole and an elastic circular inclusion under a remote uniform heat flow is investigated through observing the hoop stress along a hole boundary. By combining the Green's function derived in this paper and the existing solutions for dislocation functions, the singular integral equations for the thermoelastic problem is formulated and the unknown coefficients, which are related to the magnitude of hoop stress, are solved numerically by applying the appropriate interpolation formulae.

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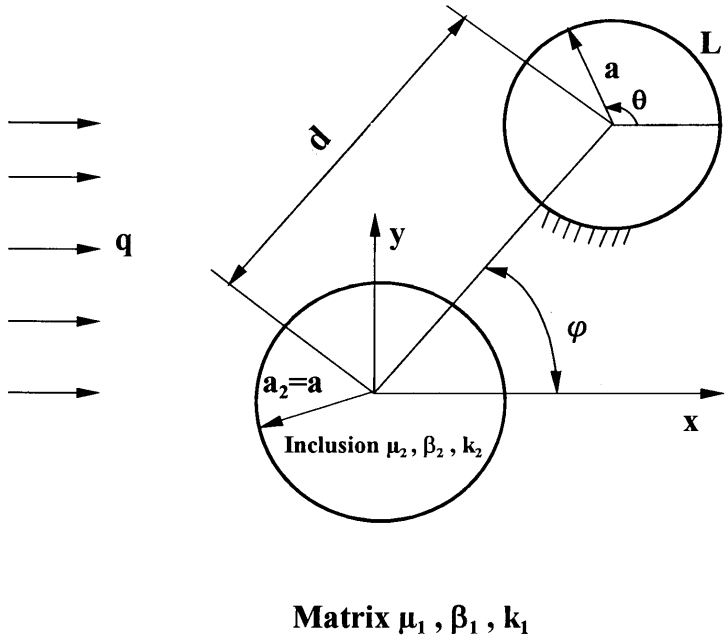

Fig. 1 A circular inclusion in an isotropic thermoelastic medium.

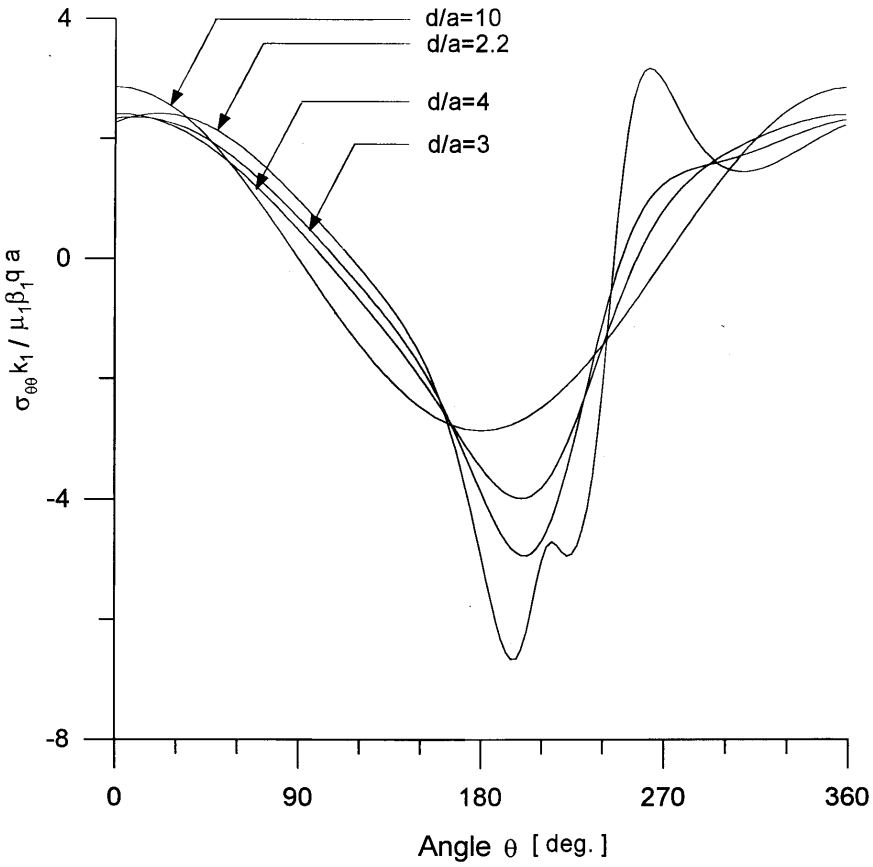

Fig. 2 Variation of hoop stress along a circular hole with $\varphi = 45^0$ and $\mu_2 / \mu_1 = \beta_2 / \beta_1 = k_2 / k_1 = 0.1\,(d / a \geq 2.2)$.

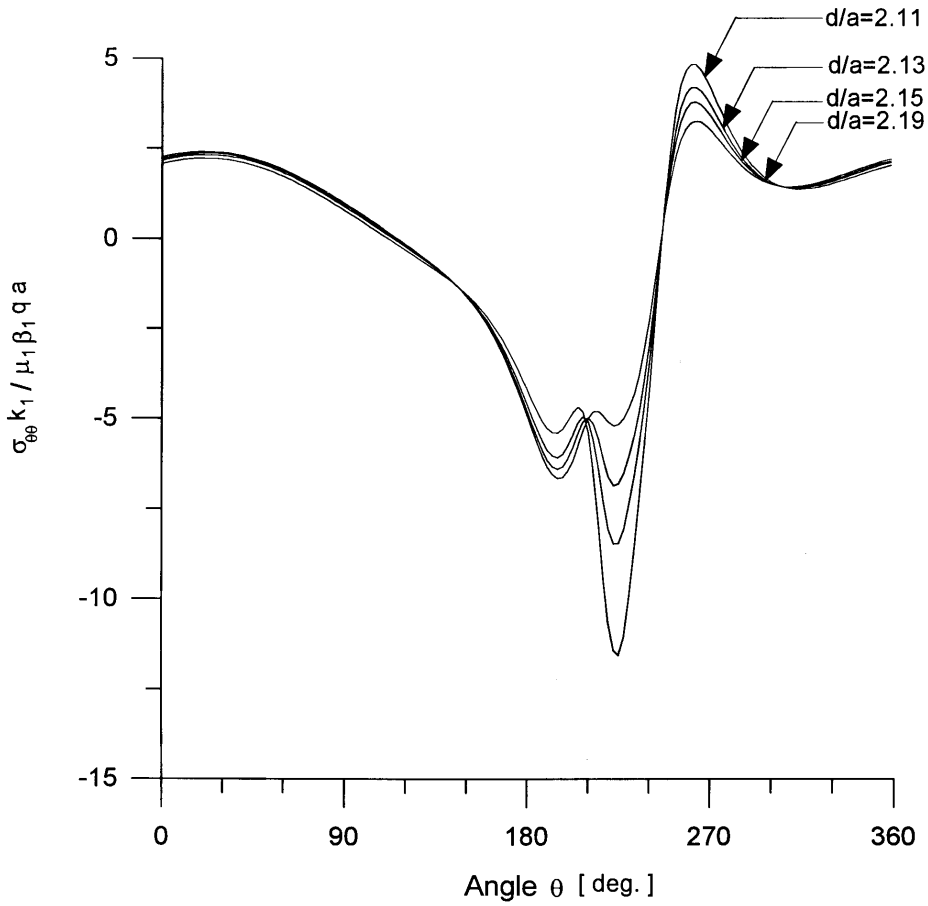

Fig. 3 Variation of hoop stress along a circular hole with $\varphi = 45^0$ and $\mu_2 / \mu_1 = \beta_2 / \beta_1 = k_2 / k_1 = 0.1$ $(d / a \leq 2.19)$.

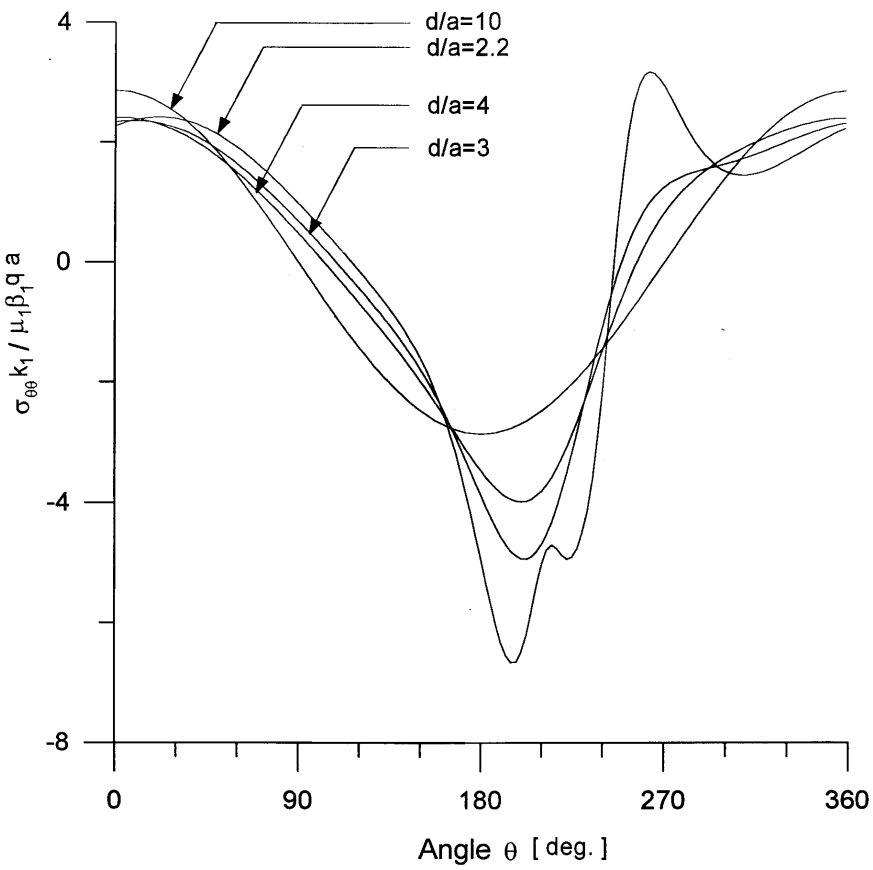

Fig. 2 Variation of hoop stress along a circular hole with $\varphi = 45^0$ and $\mu_2 / \mu_1 = \beta_2 / \beta_1 = k_2 / k_1 = 0.1\,(d / a \geq 2.2)$.