EXACT THREE-DIMENSIONAL MODELS FOR LAYERED COMPOSITES

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SUMMARY: A layered composite model based on the conceptions of the 3D theory of elasticity of anisotropic solids is proposed. This model doesn't implement traditional hypotheses about the character of the stress and strain state and is exact model for some classes of boundary problems. For obtain government equations and their solution the method of initial functions is used. Particular examples of investigation static and dynamic problems of layered thick and thin plates are considered.

KEYWORDS: 3D, anisotropy, elasticity, plate, shell, static, dynamic, thermoelasticity, electroelasticity

INTRODUCTION

As before three-dimensional linearly elastic models of composite structural members continue to draw attention of investigators. This is connected with many reasons. Among them one can state the follows. Studying of the real three-dimensional stress and strain state of the material of a composite structure enables to get useful and sufficient complete information needed for investigation the composite on the microlevel. Usually one employs some static or kinematic hypotheses that predetermine the character of the stress and strain state and make more convenient to form a model of a composite material. This causes an irreducible error in results of analysis and makes impossible to estimate correctly the strength and security of the construction.

Three-dimensional models are free from the errors related with using such hypothesis. Besides this a strong anisotropy of the composite material, a lamination, thickness heterogeneity (including continuous one), a local character and quick variety at time of loads make impossible to use one and two dimensional models of composite materials. Note also that completeness of information about the stress and strain state is never excessive in prediction of strength, rigidity and steadiness of composite structures and forming destruction models of composites. At last simple models don't enables always to represent the behavior of the material in the presence of complicated physical processes for thermo-, electro- and magnetoelasticity.

MODEL OF COMPOSITE STRUCTURAL MEMBER

In this paper for investigation of the stress and strain states of composite structural members the following model of solids is used. Consider a layered composite with arbitrary numbers of layers. Every layer can be described in a curvilinear coordinate system α, β, γ and is considered as 3D anisotropic linearly elastic solid (21 elastic constants). There are no restrictions on the thickness of layers and the structure of the laminated shell or plate. The layers are assumed to be perfectly bonded that ensures continuity of displacements and corresponding stresses when crossing layers. It is possible to have a model of the composite with continuous change of elastic constants and density of the material along one of coordinate or within separate layers. Actions on the composite may have various natures (static, dynamic, thermal, electromagnetic).

MATHEMATICAL MODEL OF COMPOSITE

Mathematical model of the layered composite is based on the equations of the theory of elasticity. The main equations of mixed method in matrix-operator form are derived using the equations of motion, Cauchy's relations and generalized Hook's law as

$$\begin{aligned} \partial_\gamma \mathbf{U}_0 &= \mathbf{D}\mathbf{U}_0 \\ \boldsymbol{\sigma} &= \mathbf{B}\mathbf{U}_0, \end{aligned} \tag{1}$$

where $\mathbf{U}_0=\{ u_\alpha \ u_\beta \ u_\gamma \ \tau_{\gamma\alpha} \ \tau_{\gamma\beta} \ \sigma_\gamma \}$, $\boldsymbol{\sigma} = \{ \sigma_\alpha \ \sigma_\beta \ \tau_{\alpha\beta} \}$, $\partial_\gamma = \partial/\partial\gamma$; **D, B** are matrices of differential operators consisting of differential operators $\partial/\partial\alpha$, $\partial/\partial\beta$, $\partial/\partial t$ of various orders, elastic constants A_{ij}, density and Lame's coefficients H_α, H_β, H_γ [1].

Using the symbolic method of integration of the differential equations (1) in power series [2, 3, 4], the main equations of the method of initial functions are obtained as:

$$\mathbf{U} = \mathbf{L}\mathbf{U}^0, \tag{2}$$

where $\mathbf{U} = \{ u_\alpha \ u_\beta \ u_\gamma \ \tau_{\gamma\alpha} \ \tau_{\gamma\beta} \ \sigma_\gamma \ \sigma_\alpha \ \sigma_\beta \ \tau_{\alpha\beta} \}$, $\mathbf{U}^0 = \{ u_\alpha^0 \ u_\beta^0 \ u_\gamma^0 \ \tau_{\gamma\alpha}^0 \ \tau_{\gamma\beta}^0 \ \sigma_\gamma^0 \}$ is a vector of initial functions (displacements and stresses) defined on the surface γ = const; $\mathbf{L} = [L_{ij}]$, $i = 1\ldots9$, $j = 1\ldots6$ is a matrix of differential operator-functions.

Also equations in the form (2) are obtained for particular cases [4, 5]:

- For the rectangular coordinate system

$$\begin{aligned} \mathbf{U} &= \{ u \ v \ w \ \sigma_z \ \tau_{yz} \ \tau_{xz} \ \sigma_x \ \sigma_y \ \tau_{xy} \}, \\ \mathbf{U}^0 &= \{ u^0 \ v^0 \ w^0 \ \tau_{xz}^0 \ \tau_{yz}^0 \ \sigma_z^0 \}; \end{aligned}$$

In this case the operator-functions are operator power series on the argument z.

- For circular cylindrical coordinate system

$$\begin{aligned} \mathbf{U} &= \{ u_z \ u_\theta \ u_r \ \tau_{rz} \ \tau_{r\theta} \ \sigma_r \ \sigma_z \ \sigma_\theta \ \tau_{\theta z} \}, \\ \mathbf{U}^0 &= \{ u_z^0 \ u_\beta^0 \ u_r^0 \ \tau_{rz}^0 \ \tau_{r\theta}^0 \ \sigma_r^0 \} \end{aligned}$$

Here the operator-functions L_{ij} are operator power series on the argument r.

The thermo-, electro- and magneto-elastic problems for composites are of special interest. For two-dimensional thermo- and electro-elastic problems the main equations of the method of

initial functions are in the form (2) with following vectors **U** и $\mathbf{U}^0$ and common designations for the temperature and the electrical potential and electroinduction [5]:
thermoelasticity

$$\mathbf{U}_T = \{\, u \;\; w \;\; T \;\; \tau_{xz} \;\; \sigma_z \;\; \sigma_x \,\},$$
$$\mathbf{U}_T^0 = \{\, u^0 \;\; w^0 \;\; T^0 \;\; \tau_{xz}^0 \;\; \sigma_z^0 \,\};$$

electroelasticity

$$\mathbf{U}_E = \{\, u \;\; w \;\; \varphi \;\; \tau_{xz} \;\; \sigma_z \;\; \sigma_x \;\; D_z \;\; D_x \,\},$$
$$\mathbf{U}_E^0 = \{\, u^0 \;\; w^0 \;\; \varphi^0 \;\; \tau_{xz}^0 \;\; \sigma_z^0 \;\; D_z^0 \,\}.$$

For layered solids the main equations of the method of initial functions are in form:

$$\mathbf{U}^k = \mathbf{L}^k \mathbf{L}_0^{k-1} \ldots \mathbf{L}_0^2 \mathbf{L}_0^1 \mathbf{U}^0, \tag{3}$$

where the matrix-column $\mathbf{U}^k$ has unknown components of the vector of stress and strain state in the k-th layer and the items of the matrices $\mathbf{L}^k$, $\mathbf{L}_0^{k-1} \ldots \mathbf{L}_0^2$, $\mathbf{L}_0^1$ are operator-functions constructing with differential operators, geometrical parameters of the layer and its elastic constants.

The equations (2) and (3) allow to create mathematical models for suitable boundary problems of the theory of elasticity.

Consider the following problem of analyzing the stress and strain state m-layered composite rectangular plate with thickness of layers h_k, elastic constants A_{ij}^k and density of layer ρ_k. The plate is supported by its bottom plane and boundary conditions on it are characterized by the vector of stresses $\mathbf{U}^+ = \{\, \tau_{xz}^+ \;\; \tau_{yz}^+ \;\; \sigma_z^+ \,\}$. The load on the upper plane of the plate are represented by the vector $\mathbf{U}_\sigma^0 = \{\, \tau_{xz}^0 \;\; \tau_{yz}^0 \;\; \sigma_z^0 \,\}$. In general it could be a function of the time argument. The governing system in partial differential operators for determining unknown initial functions, viz. the vector of displacements $\mathbf{U}_w^0 = \{\, u^0 \;\; v^0 \;\; w^0 \,\}$ on the upper plane of the plate are derived as:

$$\mathbf{L}_+^m \mathbf{L}_0^{m-1} \ldots \mathbf{L}_{0w}^1 \mathbf{U}_w^0 = \mathbf{U}^+ - \mathbf{L}_+^m \mathbf{L}_0^{m-1} \ldots \mathbf{L}_{0\sigma}^1 \mathbf{U}_\sigma^0 \tag{4}$$

Here the matrices **L** with corresponding subscripts are certain blocks of the matrices from (3) for every layer of the plate.
These equations are obtained when satisfying the boundary conditions on the bottom plane of the plate and represent in essence the exact mathematical model of the considered layered plate. From the equations (4) one can get approximate equations for various approximate theories of layered plates. For this purpose one should keep in the series for the operator-functions appropriate number of members.
When $\tau_{xz}^0 = \tau_{yz}^0 = 0$ the equations (4) are exact equations of bending of the layered plate. One can show that all known approximate theories could be derived from the equations (4).

NUMERICAL RESULTS OF ANALYSIS OF LAYERED PLATE

A numerical example is concerned with the analysis of square 5-layered plate (a x a) with h/a = 1/3, where h = 10 cm is a total thickness of the plate and a = 30 cm is a dimension of the plate side in plane. The plate is supported on its bottom plane on the contour with width 10 cm. The loads are distributed on the centered square area as shown on Fig. 1.

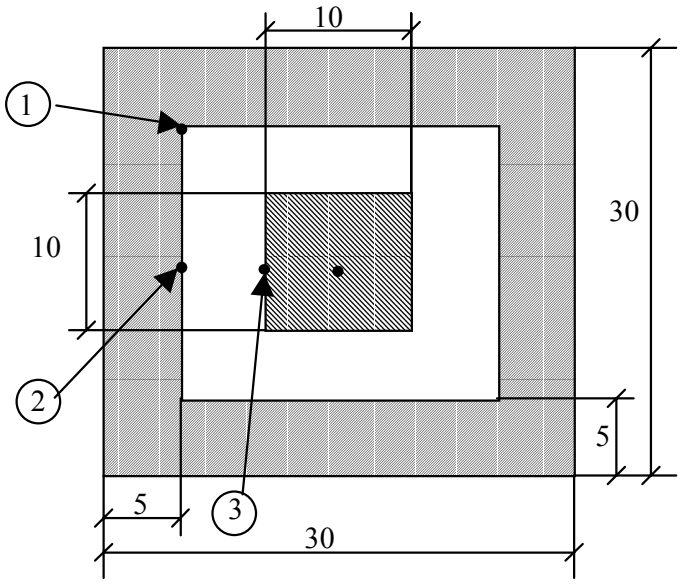

Fig. 1: Scheme of loads and geometrical dimensions of the plate

The first layer is isotropic homogeneous and fabricated from duralumin (E = 72000 *MPa*, $\nu = 0.3$, $\rho = 2.79\ 10^3\ kg/m^3$) and others are composed from micro-layers of reinforced composites with thickness 1 *mm*. Geometrical and mechanical parameters of the layers are presented in the Table 1. Densities of composites are as follows: carbon-reinforced plastic – $1.6\ 10^3\ kg/m^3$, filament-reinforced organic plastic – $1.35\ 10^3\ kg/m^3$, glass-reinforced plastic – $1.98\ 10^3\ kg/m^3$, boron-reinforced plastic – $2.04\ 10^3\ kg/m^3$.

Total number of layers are 91 ones. The load functions on the bottom and upper planes are approximated by double trigonometric series on cosines. In this case on the lateral faces of the plate the following boundary conditions are satisfied exactly:

$$\text{when } x = 0,\ a, \qquad u = 0,\ \tau_{xz} = \tau_{xy} = 0$$
$$\text{when } y = 0,\ a, \qquad v = 0,\ \tau_{yz} = \tau_{xy} = 0$$

On Fig. 2 the variation of the dimensionless stress τ_{xz}/q, where q is intensity of load, through the thickness of the plate in the section 1 (see Fig. 1) are shown. The maximum values are observed approximately in the middle of the height of the layers manufactured from duralumin, carbon-reinforced plastic and glass-reinforced plastic. In the layer of filament-reinforced organic plastic one can see the minimum of the stress approximately one third of its thickness.

The dimensionless stress σ_x/q in the section 3 is represented on Fig. 3. In this section the load changes abruptly its value. Here we can mark nonlinear character of stress in the layers of filament-reinforced organic plastic and glass-reinforced plastic. The abruptly jumps of stress value in the zone of contacting of micro-layers are observed.

Table 1: Parameters of the micro-layers

Material. Layer thickness/micro-layer/number of micro-layers mm	Fibre orientation	Elastic constants (E, G x10^3 MPa)								
		E_x	E_y	E_z	G_{xy}	G_{xz}	G_{yz}	ν_{xy}	ν_{xz}	ν_{yz}
Carbon-reinforced plastic 20/1/20	0^0	181	10.3	10.3	1.17	1.17	2.87	0.28	0.28	0.33
	90^0	10.3	181	10.3	1.17	2.87	1.17	0.016	0.28	0.33
Filament-reinforced organic plastic 50/1/50	0^0	81.3	5.41	5.41	2.03	2.03	1.7	0.34	0.02	0.35
	90^0	5.41	81.3	5.41	2.03	1.7	2.03	0.023	0.35	0.3
Glass-reinforced plastic 15/1/15	0^0	31.85	10.2	7.14	2.14	2.14	2.14	0.328	0.045	0.199
	90^0	10.2	31.85	7.14	2.14	2.14	2.14	0.105	0.139	0.201
Boron-reinforced plastic 5/1/5	0^0	241.6	15.35	15.35	5.416	5.416	4.862	0.28	0.01	0.3
	90^0	15.35	241.6	15.35	5.416	4.862	5.416	0.018	0.3	0.16

Table 2 represents natural dimensionless frequencies $\omega^* = \omega a / \sqrt{E_1 / 2\rho_1(1+\nu_1)}$ of the plate for each form of vibration *m*, *n*. Symmetric (deformations of stretching-pressuring in the transverse direction) and anti-symmetric (bending) vibrations are marked out separately.

Table 2: Natural dimensionless frequencies

m, n	Symmetric forms			Anti-symmetric forms		
1, 1	5.3448	6.1803	9.7348	1.3958	5.2767	6.1803
1, 2	2.4315	5.6264	6.2596	2.4315	5.6264	6.2596
2, 1	2.4440	5.6105	6.0428	2.4440	5.6105	6.0428
2, 2	3.1836	6.0503	10.7329	3.1836	6.0503	9.8278

The algorithm of finding natural frequencies works as follows. The deflection of upper plate surface under symmetric and anti-symmetric loads with given values *m*, *n* is calculated. These values are corresponding with the form of load (one half of the sinusoid, the whole sinusoid etc.) applied on the upper and bottom planes of the plate. At that the values of frequencies of

forced loads are changed to get the resonance state which is recognized by values of deflection of the initial plane (the upper surface).

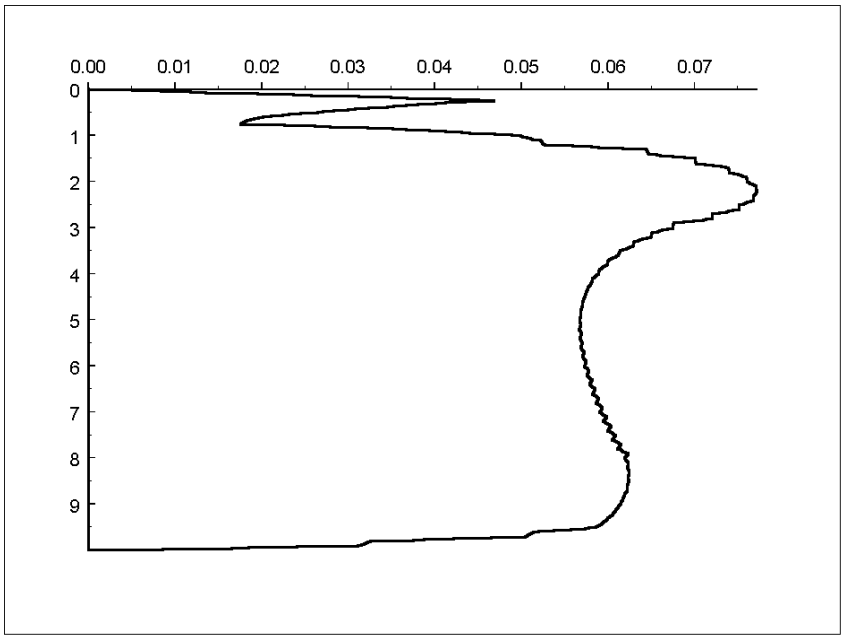

Fig. 2: Dimensionless shearing stress τ_{xz}/q in the section 1

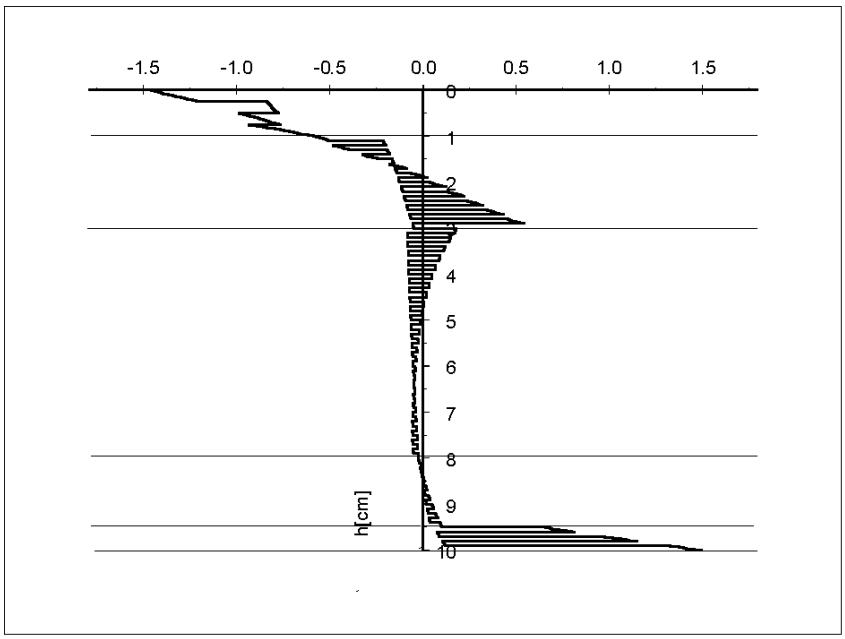

Fig. 3: Dimensionless stress σ_x/q in the section 3

A comparison of results received by the method of initial functions and experimental results [6] is presented on the Fig. 4. The plate is composed of glass fiber associated with a polyester resin. The load is a uniformly distributed on the upper surface.

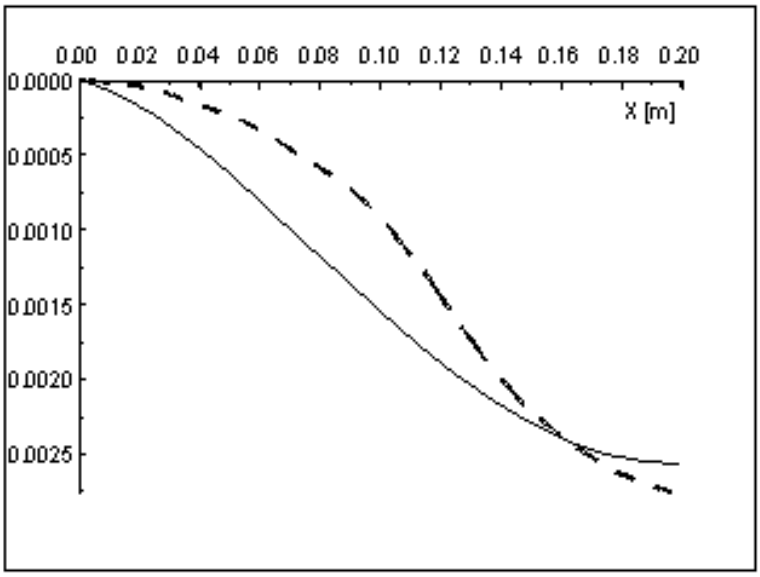

Fig. 4: Comparison of transverse displacement between an edge and the middle of the plate in the section $x=a/2$

Note that the model used in the experiment and design model differ in bounding conditions along the plate contour. In the experiment the attaching close by fixing is realized, i.e. there aren't displacements in the corresponding sections along z direction but turning angles don't equal to zero that is caused by inequality to zero of the displacements along x and y directions in each point of the section.

CONCLUSIONS

This paper shows properties of the analytical method of initial functions in analysis of layered composites on the micro-layered level. The results received are useful for designing the composite structures to evaluate the 3D stress and strain state.

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