RESONANCE CONTROL OF GENERALLY SUPPORTED SMART BEAM-PLATES USING FOURIER SERIES METHOD

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SUMMARY: This paper proposes a new method to control the resonance of composite beam-plates with generally supported boundary conditions. Firstly, the mathematical model including the general type of boundary conditions has been formulated in terms of differential and algebraic equations (DAE). Under this formulation, the natural frequencies and mode functions of the generally supported beam-plates can be computed easily. The control strategy to suppress the resonance is then conducted from two approaches: proportional control and LQG control. Some numerical examples are presented to illustrate the proposed method.

KEYWORDS: smart structure, piezo-electric, control system, Fourier series method, DAE.

INTRODUCTION

Lee and Moon [1] proposed modal sensors and actuators that the electrodes of polyvinylidence fluoride layer are trimmed in shapes according to modal functions. In asmuch as beam modal functions depend on the boundary conditions of beam-plates, the beam functions are not convenient for general applications. Wang and Lin [2] proposed the Fourier series method in conjunction with Stokes transformation which allow one to account for general types of supporting conditions. This method has been successfully applied for vibration analysis of delaminated beam-plate [3] and static shape control of smart structure [4]. In the present study, we extend the use of Fourier sine series to represent the electrode profile functions of sensors and actuators for the resonance control of composite beam-plates with generally supported boundary conditions. The matrix representation is used to overcome the algebraic cumbersome arising from the boundary conditions.

MATHEMATICAL MODEL

The following assumptions are considered in the analysis.

(1) The Bernoulli-Euler Beam theory including Kelvin-Voigt damping is used for the model.
(2) Same voltages are applied on the upper and lower actuators with opposite signs to produce pure bending moment effect on the laminated beam-plate.
(3) The electrode thickness of piezoelectric film is negligible.
(4) All laminae with the same length L and width W are bonded together perfectly.
(5) The adhesive thickness is negligible.

Based on the assumptions, the governing equation of composite beam-plates bonded with $2N_a$ laminated actuators and 2 N_s laminated sensors can be derived as follows:

$$\gamma\ddot{w}(x,t) + C_{DI}\dot{w}_{xxxx}(x,t) + Dw_{xxxx}(x,t) = f(x,t) - \sum_{a=1}^{N_a} D_a R_{a,xx} V_a(t) \tag{1}$$

where γ, D, and C_{DI} are the effective mass per unit length, bending stiffness, and Kelvin-Voigt structural damping coefficient, respectively; $D_a = (2d_{31}/3t_a)E_a W^2(z_{ua}^3 - z_{la}^3)$ in which E_a, d_{31}, t_a, z_{ua} and z_{la} are the Young's modulus, electrical strain coefficient, thickness, and distances from the middle plane to the upper and lower surfaces of the a-th actuator, respectively; V_a and R_a are the applied voltage and electrode profile function of the a-th actuator; $f(x,t)$ is the dynamic loading. The associated boundary conditions at x=0, and L are prescribed for

$$\text{either} \quad Dw_{xx} + \sum_{a=1}^{N_a} D_a R_a V_a(t) \quad \text{or} \quad w_x \qquad \text{and} \qquad \text{either} \quad Dw_{xxx} \quad \text{or} \quad w$$

The output voltage V_s of the s-th sensor is given by

$$V_s(t) = (k_{31}^2 W / C_v g_{31}) \int_0^L (z_{us} + z_{ls}) R_s(x) w_{xx} dx \tag{2}$$

where $R_s(x)$, k_{31}, g_{31}, and C_v are the corresponding electrode profile function, electromechanical coupling factor, piezoelectric stress constant, and capacitance, respectively; z_{us} and z_{ls} are the distances of the middle plane to the upper and lower surfaces.

The deflection $w(x,t)$ can be represented by a Fourier sine series as follows:

$$w(x,t) = \sum_{m=1}^{N} A_m(t)\sin(\alpha_m x) \quad \text{with} \quad A_m(t) = (2/L)\int_0^L w(x,t)\sin(\alpha_m x)dx, \quad 0 < x < L \tag{3}$$

where $\alpha_m = m\pi / L$, and N is the number of terms used in Fourier series to approximate the actual displacement. The detail relations for the w function and its derivatives expressed in terms of Fourier sine series accounting for its end values are presented in Wang and Lin [2]. The governing equation (Eqn 1) can be transformed into

$$\gamma\ddot{A}_m + C_{DI}\alpha_m^4 \dot{A}_m + D\alpha_m^4 A_m = f_m + \alpha_m(C_{DI}\dot{c}_m + Dc_m) + \sum_{a=1}^{N_a} D_a \alpha_m \overline{R}_{am} V_a \tag{4}$$

where $c_m = 2/L[(-1)^m B_1 - B_0 - \alpha_m^2((-1)^m w_1 - w_0)]$ in which $w_0 = w(0,t)$, $w_1 = w(L,t)$, $B_0 = w_{xx}(0,t)$, and $B_1 = w_{xx}(L,t)$; $\overline{R}_{am} = 2/L[(-1)^m R_{aL} - R_{a0}] + \alpha_m R_{am}$ in which R_{am} are the Fourier sine coefficient of $R_a(x)$ and $R_{a0} = R_a(0)$, $R_{aL} = R_a(L)$; f_m is the Fourier sine coefficient of $f(x,t)$. Similarly, we can obtain the corresponding equations for the boundary conditions and the sensor equation. Since only finite terms (i.e. *N*) of Fourier sine series are used in real computation, we can rewrite the governing equation, sensor output and boundary conditions into the following matrix representation

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B_1}\mathbf{v} + \mathbf{B_2}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} \tag{5}$$

where

$$\mathbf{x} = [A_1, \dot{A}_1, \cdots A_N, \dot{A}_N, w_0, w_1, B_0, B_1]^{\mathrm{T}}, \quad \mathbf{v} = [f_1, \cdots, f_N]^{\mathrm{T}},$$
$$\mathbf{u} = [V_{a1}, \cdots, V_{aN_a}]^{\mathrm{T}}, \quad \mathbf{y} = [V_{s1}, \cdots, V_{sN_s}]^{\mathrm{T}}$$

with appropriate matrices **E**(singular matrix), **A**, $\mathbf{B_1}$, $\mathbf{B_2}$ and **C**.

RESONANCE CONTROL

Resonance is controlled by adjusting the free response of a beam-plate i.e. the feedback control is used to shift the natural frequencies. The undamped natural frequency ω is determined by

$$\mathrm{Re}[\det(j\omega\mathbf{E} - \mathbf{A})] = 0, \quad j = \sqrt{-1}$$

with the mode shape computed from the corresponding eigenvector of (**E**,**A**). Suppose all states in Eqn 5 are available for feedback, let **v**=0 into Eqn 5, and consider the following linear feedback control:

$$\mathbf{u} = \mathbf{K}\,\mathbf{x} \tag{6}$$

The closed loop system becomes

$$\mathbf{E}\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B_2}\,\mathbf{K})\mathbf{x} \tag{7}$$

hence, the closed-loop undamped natural frequency ω_c is determined by

$$\mathrm{Re}[\det(j\omega_c\mathbf{E} - \mathbf{A} - \mathbf{B_2}\mathbf{K})] = 0 \tag{8}$$

with the mode shape computed from the corresponding eigenvector of $(\mathbf{E}, \mathbf{A} + \mathbf{B_2}\mathbf{K})$. One mean of selecting the matrix **K** is using proportional control technique, i.e. we relate the actuators' input voltages to the Fourier coefficients of the beam-plate deflection w and velocity $\dot{w}$ with appropriate weightings. On the other hand, we can select the matrix **K** by using optimal control approach.

Optimal Control Approach

The performance index to be minimized is given by

$$J(\mathbf{x},\mathbf{u}) = \int_0^\infty (\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u})dt \tag{9}$$

where the first term in the integrand corresponds to the state energy and the second term corresponds to the input energy. According to Katayama and Minamino [5], the optimal state feedback gain is given by

$$\mathbf{K} = -[\mathbf{U}_3 - \mathbf{R}^{-1}(\mathbf{S}^T\mathbf{V}_1 + \mathbf{B}_2^{\mathrm{T}}\mathbf{V}_2)\mathbf{H}][\mathbf{U}_1 \quad \mathbf{V}_1]^{-1} \tag{10}$$

where $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$, $\mathbf{V}_1$, and $\mathbf{V}_2$ are the solutions of the following equations:

$$\begin{aligned}
\mathbf{A}\mathbf{U}_1 - \mathbf{E}\mathbf{U}_1\mathbf{W} + \mathbf{B}_2\mathbf{U}_3 &= 0 \\
\mathbf{Q}\mathbf{U}_1 + \mathbf{A}^{\mathrm{T}}\mathbf{U}_2 + \mathbf{S}\mathbf{U}_3 + \mathbf{E}^{\mathrm{T}}\mathbf{U}_2\mathbf{W} &= 0 \\
\mathbf{S}^T\mathbf{U}_1 + \mathbf{B}_2^{\mathrm{T}}\mathbf{U}_2 + \mathbf{R}\mathbf{U}_3 &= 0 \\
\mathbf{E}\mathbf{V}_1 &= 0 \\
\mathbf{E}^{\mathrm{T}}\mathbf{V}_2 &= 0
\end{aligned}$$

and $\mathbf{W}$ is an $2N \times 2N$ Jordan form with the stable eigenvalues of $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$. The matrices $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{A}}$ are defined by

$$\tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{E} & 0 & 0 \\ 0 & \mathbf{E}^{\mathrm{T}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & \mathbf{B}_2 \\ -\mathbf{Q} & \mathbf{A}^{\mathrm{T}} & -\mathbf{S} \\ \mathbf{S} & \mathbf{B}_2^{\mathrm{T}} & \mathbf{R} \end{bmatrix}$$

RESULTS AND DISCUSSION

Verification Test

In order to verify the DAE formulation for dynamic response of flexible structures, an cantilever beam with $C_{DI} = 0$ is used as a test example. Table 1 shows the comparison between exact and computed values when 100 terms of Fourier Sine series is used:

Table 1: Comparisons between the computed and exact values for the first five natural frequencies of a cantilever beam

Modal frequency	1st	2nd	3rd	4th	5th
Exact	3.5160153	22.034492	61.697214	120.90192	199.85953
Computed(N=100)	3.5446256	22.215040	62.205935	121.90527	201.52885

At the same time, the effect of different series expansion terms on natural frequencies is investigated. As shown in Fig. 1, the computation error for modal frequencies is proportional to 1/N, e.g. if we use 100 terms in sine expansion, the relative error is less than 1%. Fig. 2 shows the effect of the number of sine expansion terms on the 2nd modal shape function. Similar result can also be obtained in analyzing the effect for modal shape functions, but more

terms in sine expansion should be used if the accuracy of modal shape functions is required to be within 1% error.

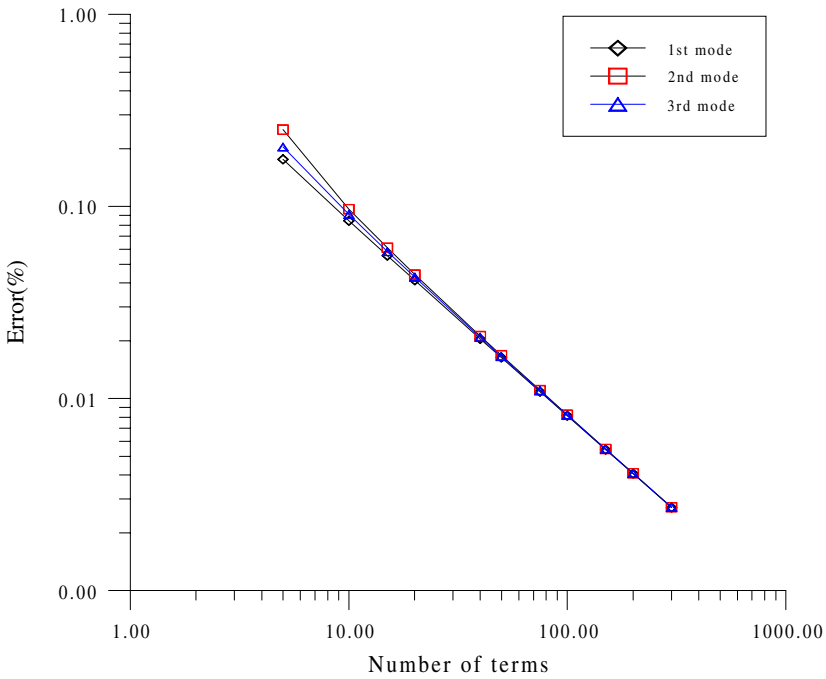

Fig. 1: The effect of sine expansion terms on errors of the natural frequencies.

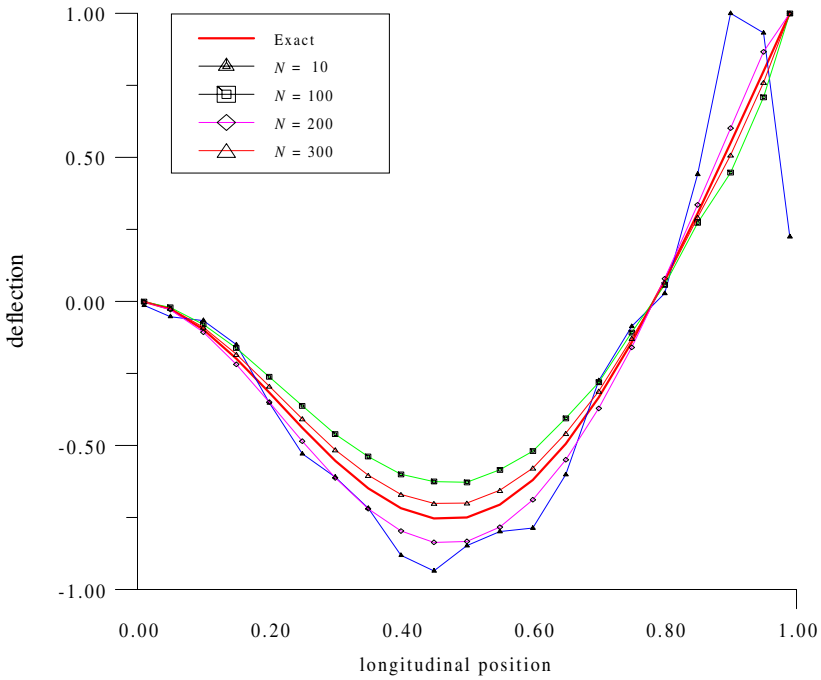

Fig. 2: The effect of sine expansion terms on the 2nd modal shape function.

Design Example

For illustrative purpose, we use a cantilever beam-plate bonded with single pair of sensors and single pair of actuators as our design example. The sensors are bounded directly to the beam-plate and the actuators are bounded on the top of the sensors. The sine shape and uniform

functions are used in the electrode profile for both sensors and actuators. In order to simplify our analysis, let $C_{DI} = 0$. Some typical results are shown in Table 2.

Table 2: Frequencies of cantilever beam-plate with different control approaches

Mode	Uniform profile function			Sine shape profile function		
	Proportion		LQG	Proportion		LQG
	$\|K\| = 10^2$	$\|K\| = 3\times10^2$		$\|K\| = 10^2$	$\|K\| = 3\times10^2$	
1	3.616	3.722	3.970	3.651	3.757	4.097
2	22.610	23.326	24.880	22.881	23.548	25.547
3	63.206	65.316	69.670	64.072	65.938	71.535

The LQG method produces a larger shift in natural frequencies. Although the Sine shape function provide better control effect than uniform function, but the improvement is not significant.

CONCLUSION

Fourier series method together with Stokes' transformation is used to represent the dynamic response of flexible structures into the Differential-Algebraic Equations form (DAE). This approach gives us more flexibility to treat generally-supported type boundary conditions. Based on our method, the computation error for modal frequencies is proportional to 1/*N*, e.g. if we use 100 terms in sine expansion, the relative error is less than 1%. Similar result in modal shape function can be obtained, but more terms in sine expansion should be used to have the accuracy of 1% error. Resonance is controlled by adjusting the free response of a beam-plate i.e. the feedback control is used to shift the natural frequencies. A cantilever beam-plate bonded with a pair of sensors and a pair of actuators is considered for illustrative purpose. The sine shape profile is more effective than uniform profile function and the LQG control strategy can further enlarge the modal frequency shift.

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