DESIGNING WITH ANISOTROPY.

PART 1 :

METHODS AND GENERAL RESULTS FOR LAMINATES.

G. Verchery.

ISAT - Institut Supérieur de l'Automobile et des Transports,
LRMA - Laboratoire de Recherche en Mécanique et Acoustique,
49, Rue Mademoiselle Bourgeois - BP 31, 58027 Nevers Cedex, France.
Georges.Verchery_isat@u-bourgogne.fr

SUMMARY : The so-called polar description for two-dimensional anisotropy is first outlined. It gives a systematic approach which generalises notions such as the well-known Mohr's circles for second order symmetrical tensors. It is an efficient tool to deal with anisotropy and consequently can be applied to analysis as well as design. Its power is illustrated in the case of the classical laminated plate theory. This theory predicts efficiently the overall behaviour of the plate in the elastic range from the properties of the laminas and the stacking sequence, but the related inverse problems, such as defining the staking sequence in order to obtain definite laminate properties, have received only limited solutions. It is shown that the use of the polar description answers to some extend to these inverse problems and introduces new concepts of composite materials. As an example, the concept of quasi-homogeneous composite is defined and the principle for designing such materials is outlined.

KEYWORDS : composites design, inverse problems, polar description of anisotropy.

INTRODUCTION.

Anisotropy has been studied for more than two centuries. Research initiated with geometrical and physical properties of crystals. For the last thirty years, anisotropy has gained a new interest with the development of composite materials, which exhibit very anisotropic characteristics. It is maybe the first time that anisotropy is connected with rather large industrial applications. Although anisotropy appears in a simplified, two-dimensional form in most of the cases, it generates real difficulties in industry, and even in research and teaching, for analysing, designing, testing, manufacturing and, purely and simply, understanding composite materials. Even though anisotropy as a theory is well-established and known in a limited circle of scientists in crystallography [1], it has not been widespread in other fields, specially among mechanical engineers. More, its practical implementation uses cumbersome formulas with many parameters. This has resulted in conservative practices, using black-box softwares or limiting rules, which is certainly detrimental to the large capabilities of these new advanced materials. Our research offers a possibility to counter these limitations.

We will describe hereafter an original presentation of two-dimensional anisotropy, that we call polar description. It is, in some respects, simpler than classical presentations, although strictly equivalent. It has been developed by the author and applied to laminate design by the author and his co-workers in various papers [2-9]. Its main advantage is that it is more powerful, as it uses more pertinent parameters. It can be applied to any field requiring anisotropy. In the following, it will be developed specially for its application to the classical laminated plate theory, on which is usually based the analysis of composite plates. We will show that it makes possible to study inverse problems of the classical laminated plate theory, which means the possibility of designing composites with required properties.
Applications are developed in Parts 2 and 3, with numerical and experimental examples. Part 2 is devoted to design of laminates without coupling between in-plane and out-of-plane elastic behaviour. Part 3 is devoted to design of the so-called quasi-homogeneous laminates

POLAR DESCRIPTION FOR TWO-DIMENSIONAL ANISOTROPY.

Descriptions of anisotropy.

A complete general description of anisotropy is obtained with the use of tensors in arbitrary co-ordinates. However, although anisotropy is one of the origins of the development of tensorial calculus, tensors are seldom utilised in usual practice. Reasons are numerous : tensors are not always known, they need some training to be mastered, tensor components may have no simple physical meaning, etc. This often results in a multiplicity of notations, competing and confusing.
In the case of anisotropic elasticity, the scalar, matrix (contracted) and tensorial notations are in use. Both matrix and tensor notations receive a compact and an index forms.
A basic problem arising in the representation of anisotropy is the transformations of equations. Such transformations occur when the reference axes are rotated or conversely, when the material is rotated. Tensorial calculus gives a straightforward, compact answer, which however is translated in very long formulas in Cartesian co-ordinates. Derivation is less direct with other notations, and gives formulas with at least the same level of complexity.
For plane anisotropy, the polar description has been derived by the author as a tensorial notation with special complex co-ordinates [2]. It will be presented hereafter in a simpler way. Although it introduces a further type of notations, it has many advantages, notably it makes very simple the transformation formulas. This polar representation generalises known concepts such as the modulus and angle for a vector, Mohr's circle construction for second order symmetrical tensors, Tsai's circles and linear combinations of components for fourth order elasticity tensors [10].

Polar representation, from vectors to elasticity tensors.

In the following, we will consider simultaneously two Cartesian frames of reference, an "old" one, with co-ordinates x_1 and x_2, and a "new" one, rotated by an angle q, with co-ordinates x and y.
The transformation formulas for these co-ordinates are :

$$x = \quad x_1 \cos q + x_2 \sin q \,, \tag{1-a}$$
$$y = -x_1 \sin q + x_2 \cos q \,. \tag{1-b}$$

Starting from vectors makes easier the understanding of polar description for plane quantities. For a vector, with Cartesian components V_1 and V_2 in the old frame, polar decomposition, with modulus R and polar angle b, is well known, as :

$$V_1 = R \cos b \,, \tag{2-a}$$
$$V_2 = R \sin b \,, \tag{2-b}$$

and the complex reverse formula is :

$$R \exp(ib) = V_1 + i\, V_2 \,. \tag{3}$$

During the change of co-ordinate axes, the Cartesian components change according to the well-known transformation, identical to Eqs. (1) :

$$V_x = \quad V_1 \cos q + V_2 \sin q \ , \qquad (4\text{-a})$$
$$V_y = -V_1 \sin q + V_2 \cos q \ . \qquad (4\text{-b})$$

The polar form is again obtained from the new components, in a form similar to Eqs. (2) and (3), with the direct formulas :

$$V_x = R\cos(b-q) \ , \qquad (5\text{-a})$$
$$V_y = R\sin(b-\theta) \ , \qquad (5\text{-b})$$

and the reverse :

$$R\exp[i(b-q)] = V_x + i\,V_y \ . \qquad (6)$$

Comparison of Eqs. (3) and (6) provides the basic rule for transformation in polar form : the modulus R is invariant and the polar angle b is decreased by the angle of rotation from the old to the new frame.
Such polar decomposition, with moduli and polar angles, can be extended to plane tensors of any order, and retains this property of very simple formulas for rotation.
The next step will be the polar description of symmetrical second order tensors, which is related to well-known concepts such as Mohr's construction or decomposition in deviatoric and spherical components. The three independent Cartesian components for such a symmetrical tensor **S** (with $S_{12} = S_{21}$) can be expressed with three polar parameters, a scalar T, a modulus D and a polar angle c, as :

$$S_{11} = T + D\cos 2c \ , \qquad (7\text{-a})$$
$$S_{12} = D\sin 2c \ , \qquad (7\text{-b})$$
$$S_{22} = T - D\cos 2c \ , \qquad (7\text{-c})$$

with the reverse equations :

$$2\,T = S_{11} + S_{22} \ , \qquad (8\text{-a})$$
$$2\,D\exp(2ic) = S_{11} - S_{22} + 2\,i\,S_{12} \ . \qquad (8\text{-b})$$

A classical result of tensor calculus is that, during the change of co-ordinate axes, the Cartesian components change to :

$$S_{xx} = S_{11}\cos^2 q + 2\,S_{12}\sin q\cos q + S_{22}\sin^2 q \ , \qquad (9\text{-a})$$
$$S_{xy} = -S_{11}\sin q\cos q + 2\,S_{12}(\cos^2 q - \sin^2 q) + S_{22}\sin q\cos q \ , \qquad (9\text{-b})$$
$$S_{yy} = S_{11}\sin^2 q - 2\,S_{12}\sin q\cos q + S_{22}\cos^2 q \ . \qquad (9\text{-c})$$

This transformation receives a much simpler form when the Cartesian components are expressed with the polar parameters :

$$S_{xx} = T + D\cos 2(c-q) \ , \qquad (10\text{-a})$$
$$S_{xy} = D\sin 2(c-q) \ , \qquad (10\text{-b})$$
$$S_{yy} = T - D\cos 2(c-q) \ , \qquad (10\text{-c})$$

whereas the polar parameters are given by :

$$2\,T = S_{xx} + S_{yy} \ , \qquad (11\text{-a})$$
$$2\,D\exp[2i(c-q)] = S_{xx} - S_{yy} + 2\,i\,S_{xy} \ . \qquad (11\text{-b})$$

So, the transformation in polar form acts according to the following simple rule : the scalar T and the modulus D are invariant and the polar angle c is decreased by the angle q of rotation from the old to the new frame.
It should be noted that the definition used for the polar angle involves a factor 2. The scalar T is the spherical part, while the modulus D is the norm of the deviatoric part of **S**. In the Mohr's

construction, the scalar T defines the position of the centre of the circle, while the modulus D is its radius.

All the above developments for vectors and symmetrical second order tensors are based on more or less known results, maybe with a somewhat different presentation and interpretation. From them, it can be guessed that this process is quite general for any order of tensors. We can give now the case of fourth order tensors with the symmetry of the elasticity tensor, i.e. complying with the following relations between their Cartesian components :

$$E_{\mathrm{ijkl}} = E_{\mathrm{ijlk}} = E_{\mathrm{jikl}} = E_{\mathrm{klij}} \,, \tag{12}$$

from which it is easily shown that such tensors have only six independent components in the two-dimensional case.

These independent Cartesian components can be expressed in the old frame as :

$$E_{1111} = \quad T_0 + 2\,T_1 + R_0 \cos 4a_0 + 4\,R_1 \cos 2a_1 \,, \qquad \text{(13-a)}$$
$$E_{1112} = \qquad R_0 \sin 4a_0 + 2\,R_1 \sin 2a_1 \,, \qquad \text{(13-b)}$$
$$E_{1122} = -\,T_0 + 2\,T_1 - R_0 \cos 4a_0 \,, \qquad \text{(13-c)}$$
$$E_{1212} = \quad T_0 \qquad - R_0 \cos 4a_0 \,, \qquad \text{(13-d)}$$
$$E_{1222} = \qquad -\,R_0 \sin 4a_0 + 2\,R_1 \sin 2a_1 \,, \qquad \text{(13-e)}$$
$$E_{2222} = \quad T_0 + 2\,T_1 + R_0 \cos 4a_0 - 4\,R_1 \cos 2a_1 \,, \qquad \text{(13-f)}$$

expressions which involve six polar parameters, i.e. two scalars T_0 and T_1, two moduli R_0 and R_1, and two polar angles a_0 and a_1.The definition of the polar angles introduces factors 4 and 2. Conversely, the six polar parameters are expressed from the six independent Cartesian components by the following four complex equations :

$$8\,T_0 = E_{1111} - 2\,E_{1122} + 4\,E_{1212} + E_{2222} \,, \qquad \text{(14-a)}$$
$$8\,T_1 = E_{1111} + 2\,E_{1122} \qquad + E_{2222} \,, \qquad \text{(14-b)}$$
$$8\,R_0 \exp(4ia_0) = E_{1111} - 2\,E_{1122} - 4\,E_{1212} + E_{2222} + 4\,i\,(E_{1112} - E_{1222}) \,, \qquad \text{(14-c)}$$
$$8\,R_1 \exp(2ia_1) = E_{1111} \qquad - E_{2222} + 2\,i\,(E_{1112} + E_{1222}) \,. \qquad \text{(14-d)}$$

From tensor calculus, the result of a rotation of co-ordinate axes is shown to be :

$$\begin{aligned} E_{\mathrm{xxxx}} = {} & E_{1111} \cos^4 q + 4\,E_{1112} \cos^3 q \sin q + 2\,E_{1122} \cos^2 q \sin^2 q \\ & + 4\,E_{1212} \cos^2 q \sin^2 q + 4\,E_{1222} \cos q \sin^3 q + E_{2222} \sin^4 q \,, \end{aligned} \tag{15}$$

and so forth for the other components.

Although linear, these formulas are cumbersome and do not allow a clear insight of the occurring process. Again, the use of the polar parameters greatly simplifies the transformation, for the Cartesian components :

$$E_{\mathrm{xxxx}} = \quad T_0 + 2\,T_1 + R_0 \cos 4(a_0 - q) + 4\,R_1 \cos 2(a_1 - q) \,, \qquad \text{(16-a)}$$
$$E_{\mathrm{xxxy}} = \qquad R_0 \sin 4(a_0 - q) + 2\,R_1 \sin 2(a_1 - q) \,, \qquad \text{(16-b)}$$
$$E_{\mathrm{xxyy}} = -\,T_0 + 2\,T_1 - R_0 \cos 4(a_0 - q) \,, \qquad \text{(16-c)}$$
$$E_{\mathrm{xyxy}} = \quad T_0 \qquad - R_0 \cos 4(a_0 - q) \,, \qquad \text{(16-d)}$$
$$E_{\mathrm{xyyy}} = \qquad -\,R_0 \sin 4(a_0 - q) + 2\,R_1 \sin 2(a_1 - q) \,, \qquad \text{(16-e)}$$
$$E_{\mathrm{yyyy}} = \quad T_0 + 2\,T_1 + R_0 \cos 4(a_0 - q) - 4\,R_1 \cos 2(a_1 - q) \,, \qquad \text{(16-f)}$$

and even much more for the polar parameters themselves, as the two scalars T_0 and T_1 and the two moduli R_0 and R_1 are invariant, while the two polar angles a_0 and a_1 are decreased by the angle of rotation q.

Finally, the reverse of Eqs. (16) are :

$$8\,T_0 = E_{\mathrm{xxxx}} - 2\,E_{\mathrm{xxyy}} + 4\,E_{\mathrm{xyxy}} + E_{\mathrm{yyyy}} \,, \qquad \text{(17-a)}$$
$$8\,T_1 = E_{\mathrm{xxxx}} + 2\,E_{\mathrm{xxyy}} \qquad + E_{\mathrm{yyyy}} \,, \qquad \text{(17-b)}$$
$$8\,R_0 \exp[4i(a_0 - q)] = E_{\mathrm{xxxx}} - 2\,E_{\mathrm{xxyy}} - 4\,E_{\mathrm{xyxy}} + E_{\mathrm{yyyy}} + 4\,i\,(E_{\mathrm{xxxy}} - E_{\mathrm{xyyy}}) \,, \qquad \text{(17-c)}$$
$$8\,R_1 \exp[2i(a_1 - q)] = E_{\mathrm{xxxx}} \qquad - E_{\mathrm{yyyy}} + 2\,i\,(E_{\mathrm{xxxy}} + E_{\mathrm{xyyy}}) \,. \qquad \text{(17-d)}$$

Further results for elasticity-type tensors obtained from their polar description.

With the use of the polar description, derivation of many properties is often eased, for theoretical problems as well as for practical ones, as exemplified hereafter.
From group theory, two-dimensional fourth order tensors with the symmetries of the tensor of elasticity possess five independent invariant functions of their six Cartesian components. A complete set of such invariants is formed by the two scalars T_0 and T_1, the two moduli R_0 and R_1, and the difference of the polar angles $(a_0 - a_1)$. The minimum degree algebraic set of six invariants constrained by one relation is formed by the two scalars $I_1 = T_0$ and $I_2 = T_1$ (degree one in the components), the square of the two moduli $I_3 = (R_0)^2$ and $I_4 = (R_1)^2$ (degree two), and the product $I_5 + i\, I_6 = R_0\, (R_1)^2 \exp[4i(a_0 - a_1)]$ (degree three), with the constraint (or syzygy) being $(I_5)^2 + (I_6)^2 = I_3\, (I_4)^2$.
The material symmetries are closely related to the polar parameters. For instance, it follows immediately from Eqs. (16) or (17) that for isotropy R_0 and R_1 must be zero.
More precisely, it can be shown that the following conditions apply :

- for orthotropy	$a_0 - a_1$ is zero or a multiple of 45°,	(18)
- for square symmetry	$R_1 = 0$,	(19)
- for isotropy	$R_0 = 0$ and $R_1 = 0$.	(20)

Consequently, the scalar parameters are called the isotropic part of the tensor, and the other parameters are its anisotropic part.
The polar description is also powerful for optimal averaging of experimental data for the determination of elastic constants of anisotropic plates [9].

Material rotation versus frame rotation.

The above development have been established for a change in the frame of reference, with no change in the position of the material of which properties are considered. When, on the contrary, the co-ordinate axes are hold while the material is rotated by an angle d, this is equivalent to a rotation of the axes by the opposite angle. So, all the previous results remain valid provided that the angle q is changed to $-d$ in all the formulas. This change in the sign of the angle is compulsory to obtain correct results, as the previous formulas use odd as well as even functions of the rotation angle.

POLAR DESCRIPTION FOR LAMINATES.

The classical laminated plate theory.

A laminate is obtained from the stacking of elementary plies. For modern composite laminates, the plies are generally anisotropic. They can be considered as elastic and in a state of plane stress. Each ply is characterised by its basic elastic properties, its specific orientation, its thickness and its position in the stacking. The ply with index k has an intrinsic plane-stress stiffness $\mathbf{Q}_k$, is oriented in the direction at an angle d_k with the first co-ordinate axis and extends from z_k to z_{k+1} along the thickness.
The well-known classical laminated plate theory [10-12] gives the following tensorial constitutive equations for general laminates in index form :

$$N_{ij} = A_{ijkl}\, e^{\circ}_{kl} + B_{ijkl}\, k_{kl} \,, \qquad (21\text{-a})$$

$$M_{ij} = B_{ijkl}\, e^{\circ}_{kl} + D_{ijkl}\, k_{kl} \,, \qquad (21\text{-b})$$

or shortly, in compact form :

$$\mathbf{N} = \mathbf{A}\, \mathbf{e}^{\circ} + \mathbf{B}\, \mathbf{k} \,, \qquad (22\text{-a})$$

$$\mathbf{M} = \mathbf{B}\, \mathbf{e}^{\circ} + \mathbf{D}\, \mathbf{k} \,. \qquad (22\text{-b})$$

In these equations, **A** is the membrane (or in-plane, or stretching) stiffness tensor, **D** is the flexural (or bending) stiffness tensor, and **B** is the coupling stiffness tensor, describing the mechanical coupling between membrane and flexure due to heterogeneity along the thickness of the plate. The generalised forces are the stress resultants $\mathbf{N}$ and the bending moments $\mathbf{M}$, while the generalised deformations are the membrane (or mean) strains $\mathbf{e}^{\mathbf{o}}$ and curvatures $\mathbf{k}$.
Further, the classical laminated plate theory expresses the stiffnesses of the laminates from the properties and stacking of the plies as :

$$\mathbf{A} = \quad \sum \mathbf{Q}^*{}_k \,(z_{k+1} - z_k)\,, \tag{23-a}$$
$$\mathbf{B} = 1/2 \sum \mathbf{Q}^*{}_k \,(z^2{}_{k+1} - z^2{}_k)\,, \tag{23-b}$$
$$\mathbf{D} = 1/3 \sum \mathbf{Q}^*{}_k \,(z^3{}_{k+1} - z^3{}_k)\,, \tag{23-c}$$

in which $\sum$ means a summation over all the plies, i.e. for the whole range of index k, and $\mathbf{Q}^*{}_k$ is the plane-stress stiffness of the ply k, obtained from the intrinsic plane-stress stiffness $\mathbf{Q}_k$ by rotation to take into account the direction at angle d_k of this ply. It should be noted that these apparently rather simple forms hide a really cumbersome algebra, as the transformation from $\mathbf{Q}_k$ to and $\mathbf{Q}^*{}_k$ involves formulas such as Eq. (15).
As a final remark, Eqs. (22) and (23) of the classical laminated plate theory hold for the thick laminated plate theory with shear effects and for the sandwich plate theory as well. In these cases, constitutive equations for transverse shear forces and transverse shear deformations are added.

Polar form of the laminated plate stiffnesses.

For each of the laminate stiffnesses **A**, **B**, and **D**, one can introduce polar parameters. As these polar parameters (in their complex form) are linear combinations in the components of the stiffnesses, Eqs. (23) are transcribed easily in polar form. So one gets the polar parameters of the laminate (with the superscript ^{m} used for membrane and the superscript ^{f} for flexure : $T_0{}^{mm}$, $T_0{}^{mf}$, etc., $F_1{}^{ff}$) in terms of the polar parameters of the plies (with the subscript $_k$ used for the ply k at angle d_k : $T_{(0)\,k}$, etc., $F_{(1)k}$) :

$$T_0{}^{mm} = \quad \Sigma\; T_{(0)\,k}\,(z_{k+1} - z_k)\,, \tag{24-a}$$
$$T_0{}^{mf} = 1/2\; \Sigma\; T_{(0)\,k}\,(z^2{}_{k+1} - z^2{}_k)\,, \tag{24-b}$$
$$T_0{}^{ff} = 1/3\; \Sigma\; T_{(0)\,k}\,(z^3{}_{k+1} - z^3{}_k)\,, \tag{24-c}$$

$$T_1{}^{mm} = \quad \Sigma\; T_{(1)\,k}\,(z_{k+1} - z_k)\,, \tag{25-a}$$
$$T_1{}^{mf} = 1/2\; \Sigma\; T_{(1)\,k}\,(z^2{}_{k+1} - z^2{}_k)\,, \tag{25-b}$$
$$T_1{}^{ff} = 1/3\; \Sigma\; T_{(1)\,k}\,(z^3{}_{k+1} - z^3{}_k)\,, \tag{26-c}$$

$$R_0{}^{mm} \exp(4iF_0{}^{mm}) = \quad \Sigma\; R_{(0)k} \exp[4i(F_{(0)k} + d_k)]\,(z_{k+1} - z_k)\,, \tag{27-a}$$
$$R_0{}^{mf} \exp(4iF_0{}^{mf}) = 1/2\; \Sigma\; R_{(0)k} \exp[4i(F_{(0)k} + d_k)]\,(z^2{}_{k+1} - z^2{}_k)\,, \tag{27-b}$$
$$R_0{}^{ff} \exp(4iF_0{}^{ff}) = 1/3\; \Sigma\; R_{(0)k} \exp[4i(F_{(0)k} + d_k)]\,(z^3{}_{k+1} - z^3{}_k)\,, \tag{27-c}$$

$$R_1{}^{mm} \exp(2iF_1{}^{mm}) = \quad \Sigma\; R_{(1)k} \exp[2i(F_{(1)k} + d_k)]\,(z_{k+1} - z_k)\,, \tag{28-a}$$
$$R_1{}^{mf} \exp(2iF_1{}^{mf}) = 1/2\; \Sigma\; R_{(1)k} \exp[2i(F_{(1)k} + d_k)]\,(z^2{}_{k+1} - z^2{}_k)\,, \tag{28-b}$$
$$R_1{}^{ff} \exp(2iF_1{}^{ff}) = 1/3\; \Sigma\; R_{(1)k} \exp[2i(F_{(1)k} + d_k)]\,(z^3{}_{k+1} - z^3{}_k)\,. \tag{28-c}$$

Although apparently complicated, these formulas (24) to (28) are in reality a tremendous simplification compared to Eqs. (23) : these Eqs. (24) to (28) include explicitly the influence of the orientations of the plies, with the angles d_k , which are hidden in Eqs. (23).

Polar form for laminates with identical plies.

A further simplification arises in the important case of laminates with all the plies made of the same material. The plies have the same material properties, i.e. the same Cartesian components as well as the same polar parameters for their intrinsic plane-stress stiffness. Of course they may have different orientations, at angles d_k.

For such laminates, material parameters in Eqs. (24) to (28) can be factorised, as they are identical. This property has already been noticed and used in classical presentation [10].
As one has, with h the total thickness :

$$\Sigma\,(z_{k+1} - z_k) = h\,, \qquad (29\text{-a})$$
$$\Sigma\,(z^2_{k+1} - z^2_k) = 0\,, \qquad (29\text{-b})$$
$$\Sigma\,(z^3_{k+1} - z^3_k) = h^3/12, \qquad (29\text{-c})$$

one gets :

$$T_0^{mm} = T_0\ h\,, \qquad (30\text{-a})$$
$$T_0^{mf} = 0 \qquad (30\text{-b})$$
$$T_0^{ff} = T_0\ h^3/12, \qquad (30\text{-c})$$

$$T_1^{mm} = T_1\ h\,, \qquad (31\text{-a})$$
$$T_1^{mf} = 0 \qquad (31\text{-b})$$
$$T_1^{ff} = T_1\ h^3/12, \qquad (31\text{-c})$$

$$R_0^{mm} \exp(4iF_0^{mm}) = R_0 \exp(4iF_{(0)k})\ \Sigma\ \exp(4id_k)\ (z_{k+1} - z_k)\,, \qquad (32\text{-a})$$
$$R_0^{mf} \exp(4iF_0^{mf}) = R_0 \exp(4iF_{(0)k})\ \Sigma\ \exp(4id_k)\ (z^2_{k+1} - z^2_k)\,, \qquad (32\text{-b})$$
$$R_0^{ff} \exp(4iF_0^{ff}) = R_0 \exp(4iF_{(0)k})\ \Sigma\ \exp(4id_k)\ (z^3_{k+1} - z^3_k)\,, \qquad (32\text{-c})$$

$$R_1^{mm} \exp(2iF_1^{mm}) = R_1 \exp(2iF_{(1)k})\ \Sigma\ \exp(2id_k)\ (z_{k+1} - z_k)\,, \qquad (33\text{-a})$$
$$R_1^{mf} \exp(2iF_1^{mf}) = R_1 \exp(2iF_{(1)k})\ \Sigma\ \exp(2id_k)\ (z^2_{k+1} - z^2_k)\,, \qquad (33\text{-b})$$
$$R_1^{ff} \exp(2iF_1^{ff}) = R_1 \exp(2iF_{(1)k})\ \Sigma\ \exp(2id_k)\ (z^3_{k+1} - z^3_k)\,. \qquad (33\text{-c})$$

These formulas show important properties of these laminates made of identical plies. The following results have been partly unnoticed before, because the complexity of the general Eqs. (23) hides them.
First, the isotropic part of such laminates for membrane and flexure is equal to the isotropic part of the basic ply. Further, there is no coupling in the isotropic part. As a practical consequence, the so-called quasi-isotropic laminates, which are designed to be isotropic for membrane properties and have received a considerable attention for aeronautical applications, have membrane stiffnesses equal to the isotropic part of their constitutive plies.
Secondly, the anisotropic parts are expressed as the product of the anisotropic part of the ply by factors completely defined by the stacking sequence. The absolute values of these factors are always less or equal to one, and generally decrease when the number of plies increases. This means that generally, when the number of plies increases, the laminate tends to resemble a homogeneous material with the properties of the isotropic part of the basic ply.
In current applications, identical plies have the same thickness h/n, with n the total number of plies. Then the laminate is completely defined by its stacking sequence, with the list of the angles (commonly given in degrees) from the bottom of the laminate to the top, as follows :

$$[d_1/d_2/d_3/\ ....\ /d_{n-1}/d_n] \text{ or } [d_{-p}/d_{-p+1}/d_{-p+2}/\ ....\ /d_{p-1}/d_p]$$

To proceed further, it is necessary to distinguish between the odd and even values of the number of plies. For any quantity f which takes a constant value f_k in the ply of index k, let us introduce the following linear symbolic operators :

$$A(f) = 1/n\ [\,f_0 + \Sigma\ (f_k + f_{-k})\,] \quad \text{for odd } n, \qquad (34\text{-a})$$
$$A(f) = 1/n\ \Sigma\ (f_k + f_{-k}) \quad \text{for even } n, \qquad (34\text{-b})$$

$$B(f) = \sqrt{3}/n^2\ \Sigma\ 2k\ (f_k + f_{-k}) \quad \text{for odd } n, \qquad (34\text{-c})$$
$$B(f) = \sqrt{3}/n^2\ \Sigma\ (2k - 1)\ (f_k - f_{-k}) \quad \text{for even } n, \qquad (34\text{-d})$$

$$D(f) = 1/n^3\ [f_0 + \Sigma\ (12k^2 + 1)(f_k + f_{-k})] \qquad \text{for odd } n, \qquad (34\text{-e})$$
$$D(f) = 1/n^3\ \Sigma\ 4(3k^2 - 3k + 1)(f_k + f_{-k}) \qquad \text{for even } n. \qquad (34\text{-f})$$

Here the index of plies runs from $-p$ to -1 and $+1$ to $+p$ for odd $n = 2p$, or from $-p$ to $+p$, including 0, for even $n = 2p+1$, while the summations Σ are from 1 to p.
With these operators, Eqs. (32) and (33) rewrite :

$$R_0^{mm} \exp(4iF_0^{mm}) = R_0 \exp(4iF_{(0)k})\ A(\exp 4id)\ h, \qquad (35\text{-a})$$
$$R_0^{mf} \exp(4iF_0^{mf}) = R_0 \exp(4iF_{(0)k})\ B(\exp 4id)\ h^2/2\sqrt{3}, \qquad (35\text{-b})$$
$$R_0^{ff} \exp(4iF_0^{ff}) = R_0 \exp(4iF_{(0)k})\ D(\exp 4id)\ h^3/12, \qquad (35\text{-c})$$

$$R_1^{mm} \exp(2iF_1^{mm}) = R_1 \exp(2iF_{(1)k})\ A(\exp 2id)\ h, \qquad (36\text{-a})$$
$$R_1^{mf} \exp(2iF_1^{mf}) = R_1 \exp(2iF_{(1)k})\ B(\exp 2id)\ h^2/2\sqrt{3}, \qquad (36\text{-b})$$
$$R_1^{ff} \exp(2iF_1^{ff}) = R_1 \exp(2iF_{(1)k})\ D(\exp 2id)\ h^3/12, \qquad (36\text{-c})$$

In the following, we will also make use of the difference operator :

$$C(f) = A(f) - D(f), \qquad (37)$$

and of special values, when all the f_k have the same value $f_k = 1$ all along the thickness :

$$A(1) = 1, \qquad (38\text{-a})$$
$$B(1) = 0, \qquad (38\text{-b})$$
$$C(1) = 0, \qquad (38\text{-c})$$
$$D(1) = 1, \qquad (38\text{-d})$$

and when all the f_k are multiplied by the same constant $\boldsymbol{a}$:

$$A(\boldsymbol{a} f) = \boldsymbol{a} A(f), \qquad (39\text{-a})$$
$$B(\boldsymbol{a} f) = \boldsymbol{a} B(f), \qquad (39\text{-b})$$
$$C(\boldsymbol{a} f) = \boldsymbol{a} C(f), \qquad (39\text{-c})$$
$$D(\boldsymbol{a} f) = \boldsymbol{a} D(f). \qquad (39\text{-d})$$

DESIGN OF LAMINATES.

A critical review on design of laminates was presented by the author in a previous work [8] and is sum-marized hereafter.
Only few papers have been published for the design of stacking sequence, specifying some lamination arrangement conditions. Some are general tools and others are for specific results only. The oldest example of general rules is certainly the well-known conditions published by Werren and Norris in 1953 [13] for the so-called quasi-isotropic laminates, i.e. isotropic for in-plane properties. The rule of mid-plane symmetry of the stacking sequence to eliminate stretching-bending coupling, as presented by Bert together with various specific cases, cannot be certainly dated and credited [14]. Other works have given special rules or results [15-19].
Not only these rules are sometimes limited in their aim, but they are often misunderstood. The point is that they are generally misconceived as necessary conditions, although they are only sufficient to reach the required properties. It should be kept in mind, for sake of exactness and to make possible future progress in design, that it is common in design methodology to rely on sufficient conditions to get solutions. That is the case with these rules and results that we will present in the following have the same character. Currently used rules for designing stacking sequences are very restrictive and could impede the development of composite materials. For instance, it is a pity that in the present practice, mid-plane symmetry is the only solution used to suppress the stretching-bending coupling, in spite of the works of Caprino & Crivelli-Visconti [15] and of Vong & Verchery [3], in which non-symmetric uncoupled laminates were presented.

The small number and the limited power of the current design rules certainly originate in the fact that no method was available to easily handle the many parameters of anisotropy and lamination theory. These rules were consequently obtained more or less by chance. Polar representation of anisotropy being a powerful tool to master these parameters, the author and his co-workers have been able to investigate systematically the inverse problems of the classical laminated plate theory [3-8]. It has resulted in new rules, new methods and new concepts of materials. So, rules for laminates without membrane-flexure coupling or laminates isotropic in flexure were presented. Quasi-homogeneous laminates were defined and illustrated by examples. The concept of norm and distance between tensors was introduced and used to define materials which satisfy approximately some criterion, so-called nearly isotropic, nearly homogeneous, etc.

This is outlined in the following with the case of quasi-homogeneous laminated plates, which is described in full details in Part 3, while Part 2 presents the case of laminates without membrane-flexure coupling.

DESIGN OF QUASI-HOMOGENEOUS LAMINATES.

Definition of quasi-homogeneous laminates.

We define a quasi-homogeneous laminate as a laminate with no membrane-bending coupling and equal normalised stiffnesses for membrane and bending :

$$\mathbf{A} / h - \mathbf{D} /(h^3/12) = \mathbf{0} \,, \tag{40-a}$$

$$\mathbf{B} = \mathbf{0} \,, \tag{40-b}$$

Such laminates do exist, as shown by the trivial case of a laminate in which all the plies have the same material properties and direction. More interesting examples have been published in our previous works. These quasi-homogeneous laminates are similar to a plate made of homogeneous material, at least from the point of view of stiffness. They have a really simpler behaviour than general laminates and many results available for usual materials from classical methods or textbooks apply to them. Further, they can have even more particular properties, such as orthotropy or isotropy. So they offer great potentialities for practical applications. However, they have not been introduced or even noticed prior to our work and of course no rule to build such materials has been proposed.

Designing such laminates need to attack the inverse problem defined by Eqs. (40). No simple results are obtained for laminates with arbitrary plies. In the following, we will concentrate on laminates made of plies with the same material properties, as it is generally done for design rules.

Conditions for quasi-homogeneous laminates with identical plies.

The above tensorial Eqs. (40) represent a total of 12 real scalar equations if one uses the Cartesian components of the tensors, or equivalently, 4 real scalar equations and 4 complex scalar equations if one uses the polar components.

When using these polar components, a simplification appears for laminates made of plies with the same material properties, for the isotropic parts of the membrane and bending stiffnesses are identical, while the isotropic part of the coupling stiffnesses vanishes, as shown by Eqs. (30) and (31). It should be emphasised that this intrinsic simplification in the constitutive equations remains unnoticed when using Cartesian components.

So the conditions for quasi-homogeneity reduce to 4 complex scalar equations, obtained by setting to zero the complex anisotropic parts of the tensors and making use of Eqs. (35) to (37) :

$$B(\exp 4 i d) = 0 \,, \tag{41-a}$$

$$B(\exp 2 i d) = 0 \,, \tag{41-b}$$

$$C(\exp 4 i d) = 0 \,, \tag{41-c}$$

$$C(\exp 2 i d) = 0 \,, \tag{41-d}$$

This is a set of four complex transcendent equations in the n unknowns $\boldsymbol{d}_k$. Changing to the variables $t_k = \tan \boldsymbol{d}_k$ and separating real and imaginary parts, it can be transformed in a set of 8 real algebraic

equations in the n unknowns t_k . As a general demonstration of existence and number of solutions has not been obtained and is out of the scope of this paper, we will give only limited discussion.
First, solutions do exist for any value of n, as one has, from Eqs. (42), $B(1) = 0$ and $C(1) = 0$, which gives the trivial solution $d_k = 0$ for all k, with all the n plies in the same direction. Further, it results from Eqs. (43) that solutions are determined except from an additive constant, i.e. when the d_k are a solution, an infinity of other solutions is generated by $d_k + g_0$, with g_0 an arbitrary angle.
When a solution with angles d_k has been found for n plies, one to three distinct solutions are readily obtained for $2n$ plies, namely one by repeating twice each ply in the sequence, the others by repeating twice the whole stacking sequence in the same and opposite orders. These solutions are generally distinct. This principle can be extended for $2n$, $3n$, etc. plies and generally to combinations of known solutions to get new solutions.
Up to $n = 5$, we succeeded in solving completely the equations and so have shown that the trivial solutions are the only ones. For larger n, we developed special solutions, that we called quasi-trivial solutions, which are described in Part 3. It should be noticed that we got non trivial solutions for $n = 7$, although the number of unknowns is then less than the number of equations, while we failed to find non trivial solutions for $n = 9$ or 10, although in these cases the number of unknowns is greater than the number of equations.
Further, it can be shown that the laminates with these quasi-trivial stacking sequences are homogeneous, not only for elastic properties, but also for thermal expansion. So their whole thermoelastic behaviour is similar to that of an homogeneous material. Specially, they have no thermal coupling, which allows manufacturing through hot pressing without introducing geometrical distortion.
Finally, it should be emphasised that only stacking sequences are required to define quasi-trivial solutions. So they are valid whatever the material of the plies is, giving large possibilities for practical applications or allowing to prescribe further design conditions.

CONCLUSION.

The polar description of two-dimensional anisotropy has been presented, for vectors, second order symmetrical tensors and fourth order elasticity tensors. As it is more powerful and somewhat simpler than classical descriptions of anisotropy, it makes possible the treatment of inverse problems. This has been illustrated in the field of composite materials. With polar parameters, the equations of the classical laminated plate theory rewrites in a clearer, more efficient form, which allows the definition of new concepts of laminates such as the quasi-homogeneous material. Parts 2 and 3 presents and illustrates, from theoretical and experimental results, the efficiency of the polar description in the special important cases of laminates without coupling and quasi-homogeneous laminates.

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