OPTIMUM DESIGN OF GRID CYLINDRICAL STRUCTURES USING HOMOGENISED MODEL

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SUMMARY: A new homogenisation approach was applied for the analysis of composite isogrid structures. This approach allows to make the optimisation of the isogrid structures faster. The mathematical model has been verified by means of the comparative analysis for the homogenised model and exact FEM model. Results of the analysis are presented and discussed.

KEYWORDS: Optimisation, composite structures, design, homogenisation, FEM.

INTRODUCTION

Structures for different engineering applications must satisfy a variety of functional parameters and properties depending on their usage. Composite isogrid structures provide a great potential to replace conventional metal structures by offering higher strength to weight ratio, flexibility in design, custom tailoring of desired properties and the ability to sustain different environments. Many distinct areas of application for composite isogrid structures have emerged. These are payload shrouds, solar array substrates and civil engineering structures. Due to the luck of automation or modular construction, they are expensive to manufacture. Only within recently the concept of the grid made of unidirectional continuous fibres was introduced and it is predominantly applied in aerospace industry. This structural concept also has a great potential for automation of manufacturing process including continuous filament winding and resin transfer moulding.

The concept of isogrid represents the lattice of interconnected ribs that made from continuous very strong, stiff and tough fibres. The ribs that comprise the structure are arrange in three families in the way that creates a repetitive pattern of triangular cell (Figure 1).

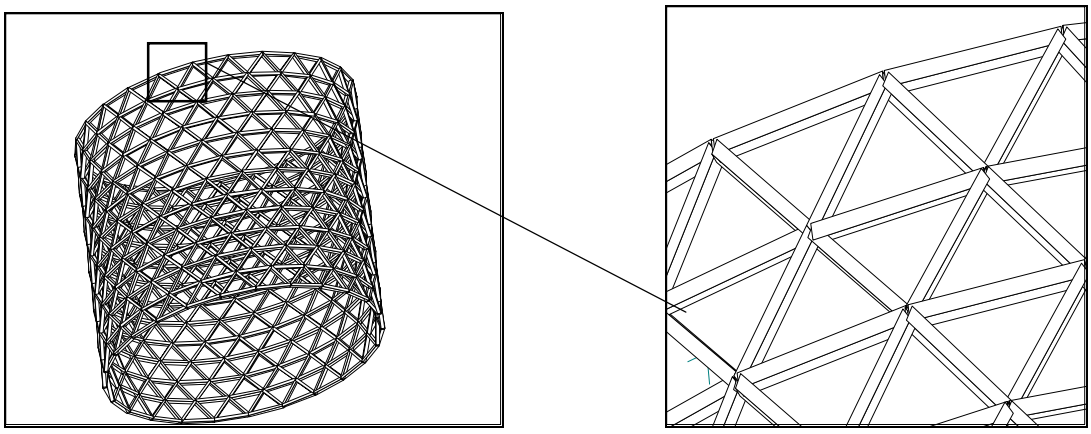

Figure 1 Isogrid structure

Unidirectional arrangement of the ribs possesses good impact damage tolerance, resistance to delimination and crack propagation across the grid. The first generation of isogrid structures was metal isogrids, which were essentially the integral equilateral triangular stiffening ribs machined onto a metallic skin surface. Manufacturing of such metal isogrids was very laborious task and subsequently required up to two years of lead time. Besides it was an expensive procedure and obtained components came out excessively heavy. Significant reduction of weight and subsequently building costs was achieved by replacing the aluminium shrouds structure with those made of the composite materials.

In contrast to the laminates, grid structures posses multidirectional stiffness by running the ribs in several directions, which allows to avoid material mismatch associated with laminates. The luck of material mismatch provides composite grid structures with much better resistance to delimination, crack propagation and superior impact damage tolerance (the later is usually occurs at the interface between different material layers in laminates).

The ribs comprising a grid structure are usually loaded in their axial directions so that in composite grid structures fibres are usually oriented along the rib's axial direction which allows to provide their maximum axial strength and stiffness. Since composite ribs are orthotropic their transverse stiffness is much less than axial. In order to compensate the lost of transverse (bending) stiffness the ribs are usually maintain high aspect ratio in that direction (from 3 to 5). Inplane stiffness of grid structures can also be altered subjected to the number of families of ribs incorporated and their mutual orientation which will subsequently define a repetitive cell pattern.

Isogrid with its triangular cell pattern exhibits high strength for both normal and shear loading. But despite its name the isogrid is not isotropic and shows much higher strength in the direction coincident with axial ribs than in the direction transverse to it. When the aspect ratio of the rib's cross-section increases (it becomes taller and thinner) the buckling of the ribs becomes an important factor, isogrid structures exhibit significantly higher strength to tensile loads than to compressive loads. Isogrids are often become weaker and tend to axial rib's buckling when subjected to the combined load: compressive along the axial ribs and tensile transverse to the axial ribs. However, in general composite isogrid significantly outperforms the aluminium

one due to the ability to run composite fibres exclusively axially along the ribs, which tremendously increases their strength.

Composite isogrid structures can offer a wide variety of properties for different sequences of geometric parameters (rib cross-section, skin thickness, configuration of the unit cell) and skin lay-ups. All these advantages in the design can only be utilised by incorporating the optimisation methods.

During optimisation process the design responses of the structure are usually calculated from the stress resultants in the structural members. Calculation of the stress resultants themselves in the isogrid using finite element method can be significantly simplified by means of incorporating homogenise model to substitute exact FEM model. Transition from the exact model to equivalent stiffness model (ESM) is developed and verified in [1], where the structure was represented by means of equivalent stiffness matrices $[A]$, $[B]$, $[D]$ and $[H]$ that describe extensional, bending-coupling, flexural and transverse shear stiffness. Design-optimisation process usually requires performing a great number of finite element analyses for the constantly changing optimisation variables. In this case, equivalent stiffness matrices $[A]$, $[B]$, $[D]$ and $[H]$ have to be each time recalculated to update the stiffness characteristics of the finite element model. Calculation of the equivalent stiffness matrices of the isogrid is not a part of the FEM code so they should be calculated manually and subsequently the FEM model has to be manually updated. Considering the amount of optimisation iterations required in order to obtain final optimum design with the reasonable convergence tolerance, this task is obviously extremely extensive. Besides, the number and type of design variables that can be possibly assigned in now commercially available FEM packages is limited. That is why the optimisation of the isogrid structures must be performed using optimisation routine that is not based on the FEM calculation of the design responses. In the present study the approach is developed that does not require performance of the FEM analysis. The different homogenisation principles were used for the formulation of ESM.

HOMOGENISED MODEL

The proposed approach deals with the elastic shell as a continuous system, i.e. external loads and the stress-strain state is described by the functions of constantly changing arguments. This approach allows effective implementation of the methods of solid mechanics in the analysis of isogrid shell.

Most of the grid shells represent a complex spatial frames comprised of elastic members. The axes of the structural members (the ribs) are assumed to form families of curves on the median surface of the shell. These families will be further referred as “families of ribs”. There are three families for the isogrid structure (Figure 2).

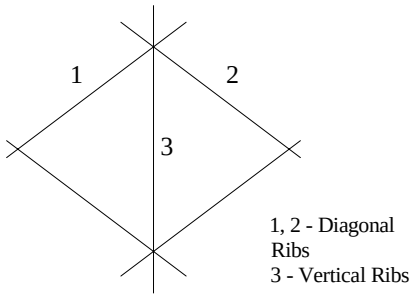

Figure 2 Unit cell

In the approach presented the composite structure in the form of the shell of revolution is considered (Figure 3).

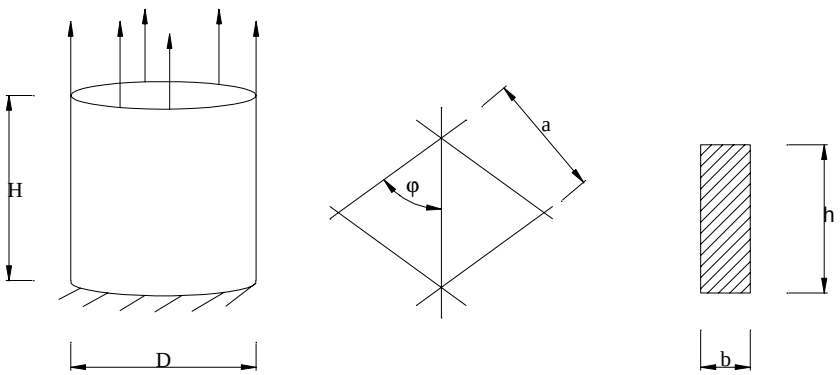

Figure 3 Considered structure and cell pattern

The median surface of the continuous model assumed to be coincident with the one of the exact model. Axis of the ribs that belong to any particular family do not intersect. The accuracy of the calculation is high except the areas which are in the close vicinity to the boundary. For these regions boundary effect must be taken into account or calculations should be used on the basis of the exact model. Three groups of equations are employed:

1. Known static equations and equations of motion written in terms of forces and moments
2. Geometric equations that link the displacements to the deformations
3. Constitutive equations of the shell (1).

$$
\begin{aligned}
&N_1^0 = \alpha_{11}\varepsilon_1 + \alpha_{12}\varepsilon_2, \quad N_2^0 = \alpha_{12}\varepsilon_1 + \alpha_{22}\varepsilon_2, \quad S^0 = \alpha_{12}\omega, \\
&M_1^0 = \gamma_{11}\chi_1^0 + \gamma_{12}\chi_2^0, \quad M_2^0 = \gamma_{12}\chi_1^0 + \gamma_{22}\chi_1^0, \\
&H_1^0 = \gamma_{31}\tau^0, \qquad H_2^0 = \gamma_{41}\tau^0.
\end{aligned} \tag{1}
$$

The above mentioned equations after some transformations yield to the system of differential equations that can be written in following normal form:

$$
y'\left(z^0\right) = P\left(z^0\right)y\left(z^0\right) + f\left(z^0\right), \tag{2}
$$

where y is the vector of unknown displacements $\left(u^0, w^0, \gamma_1\right)$, forces $\left(N_1^0, Q_1^0\right)$ and moment $\left(M_1^0\right)$ acting in homogenise model:

$$
y_1 = u^0, \;\; y_2 = w^0, \;\; y_3 = \gamma_1, \;\; y_4 = N_1^0, \;\; y_5 = M_1^0, \;\; y_6 = Q_1^0, \tag{3}
$$

f is a known vector describing loading conditions of the shell:

$$
f_1 = f_2 = f_3 = f_5 = 0, \qquad f_4 = -AX, \qquad f_6 = -AZ, \tag{4}
$$

P is a square matrix with known coefficients that are calculated from the stiffness properties and geometric characteristics of the given shell:

$$
p_{11} = -\frac{\alpha_{12}B^{0'}}{\alpha_{11}B^0}, \qquad p_{12} = -\frac{\alpha_{12}mA}{\alpha_{11}B^0}, \qquad p_{13} = -A\left(k_1^0 + \frac{\alpha_{12}}{\alpha_{11}}k_2^0\right), \qquad p_{21} = -\frac{mA}{B^0},
$$

$$
p_{22} = \frac{B^{0'}}{B^0}, \qquad p_{31} = Ak_1^0, \qquad p_{41} = \frac{\gamma_{11}\psi_0}{\gamma_{12}} + \frac{\alpha_{12}k_1^0B^{0'}}{\alpha_{11}B^0}, \qquad p_{34} = A,
$$

$$
p_{42} = \frac{\alpha_{12}mk_1^0A}{\alpha_{11}B^0}, \qquad p_{44} = -\frac{\gamma_{12}B^{0'}}{\gamma_{11}B^0}, \qquad p_{43} = A\left(\frac{\gamma_{12}\psi_1}{\gamma_{11}} + \frac{\alpha_{12}}{\alpha_{11}}k_1^0k_2^0\right), \; p_{45} = -\frac{k_1^0A}{\alpha_{11}},
$$

$$
p_{51} = \frac{1}{B^{0^2}}\left(\alpha_0\frac{\left(B^{0'}\right)^2}{A} + m^2k_1^0k_2^0\gamma_{31}A\right), \qquad p_{52} = m\alpha_0\frac{B^{0'}}{B^{0^2}}, \qquad p_{62} = \left(\alpha_0 - \psi_3\right)\frac{m^2\psi_2A}{B^{0^2}},
$$

$$
p_{56} = -\frac{mA}{B^0}\left(1 + \frac{k_1^0\gamma_{31}\left(k_1^0 - k_2^0\right)}{2\alpha_{12}}\right), \qquad p_{53} = \frac{B^{0'}}{B^0}\left(\alpha_0k_2^0 - \frac{\gamma_{31}}{B^{0^2}}m^2k_1^0\right),
$$

$$
p_{61} = \frac{m\psi_2}{B^{0^2}}\left[\left(\alpha_0 - \psi_3 - \beta_1k_2^{0^2}\right)B^{0'} - \left(\beta_0 + \beta_2\gamma_{31}\right)\psi_0k_2^0B^0 - \beta_3k_2^{0^2}\right],
$$

$$
p_{63} = \frac{m\psi_2k_2^0A}{B^0}\left[\begin{aligned}
&\alpha_0 + \frac{\beta_1\left(B^{0'}\right)^2}{A^2B^{0^2}} + \frac{\gamma_{31}\left(B^0\right)''}{A^2B^0} - \psi_1\left(\beta_0 + \frac{\gamma_{12}\gamma_{31}}{\gamma_{11}}\right) + \\
&k_2^0\gamma_{31}\left[k_1^0 + \left(k_2^0 - k_1^0\right)\frac{\alpha_{12}}{\alpha_{11}}\right] + \frac{B^{0'}}{A^3B^{0^3}}\left[\gamma_{31}{}'AB^{0^2} - \gamma_{31}B^0\left(A'B^0 + 2AB^{0'},\right)\right]
\end{aligned}\right],
$$

$$p_{64} = \frac{m\psi_2 k_2^0}{B^{0^2}}\left[\left(\beta_0 - \beta_1 + \beta_2\gamma_{31}\right)B^{0'} - \beta_3\right],\ p_{65} = \frac{m\psi_2 A}{\alpha_{11}B^0}\left[\alpha_{12}\left(k_1^0 - k_2^0\right)k_2^0\gamma_{31}\right],$$

$$p_{66} = -\psi_2\left[\frac{2B^{0'}}{B^0} + \left(k_2^{0'} - k_1^{0'}\right)\frac{k_2^0\gamma_{31}}{2\alpha_{12}} + \frac{k_2^0\left(k_2^0 - k_1^0\right)}{2\alpha_{12}}\left[\frac{\beta_1 B^{0'}}{B^0} + \left(\gamma_{31}' - \frac{\gamma_{31}\alpha_{12}'}{\alpha_{12}}\right)\right]\right]. \quad (5)$$

Nonzero members of matrix P (equation (5)) are functions of stiffness characteristics of a grid material, geometrical and cross-sectional parameters of a cell and structure as a whole. In the case of axisymmetric loading coefficients of the matrix P will have simplified form.

Solution of the system of differential equations (2) must satisfy the following boundary conditions:

$$B_0 y\left(z^0\right) = b_0 \text{ for } z^0 = z_0^0, \qquad B_1 y\left(z^0\right) = b_1 \text{ for } z^0 = z_1^0. \quad (6)$$

The given system of differential equations (2) represents a boundary value problem that can be solved numerically reducing it to the solution of Cauchy problem using method of Runge-Kutta.

The rest of the unknowns which are not in the system (2) have to be calculated using the following formulae:

$$N_2^0 = \left(\alpha_{22} - \frac{\alpha_{12}^2}{\alpha_{11}}\right)\left(\frac{B^{0'}}{AB^0}u^0 + k_2^0 w^0\right) + \frac{\alpha_{12}}{\alpha_{11}}N_1^0,$$

$$M_2^0 = \left(\gamma_{22} - \frac{\gamma_{12}^2}{\gamma_{11}}\right)\left(\left(k_2^{0'} - \frac{B^{0'}}{B^0}k_1^0\right)\frac{u^0}{A} - k_2^{0^2}w^0 - \frac{B^{0'}}{AB^0}\gamma_1\right) + \frac{\gamma_{12}}{\gamma_{11}}M_1^0. \quad (7)$$

MODEL VERIFICATION

The isogrid structure shown on the Figure 3 was analysed. Considered structure consists of three families of ribs which create a pattern of equilateral triangular cell: repetitive structural unit of the isogrid. The structure has a form of a circular cylinder that is fixed on the one end and loaded with the uniformly distributed tensional load of 10^6 N/m along the opposite end. The dimensions of the structure (Figure 3) are:
H=7.56 m; D=5.44 m; φ=60°; a=0.302 m; h=0.02 m; b=0.00666667 m. Material properties are given in the Table 1.

Material	E_1, Pa	E_2, Pa	ν_{12}	G_{12}, Pa
T300/5208	$1.81\ 10^{11}$	$1.03\ 10^{10}$	0.28	$7.17\ 10^{9}$

Table 1 Material properties of the ribs and skins.

On the basis of equations (2) analysis of homogenised mathematical model was performed. Computer routine for calculation of equivalent homogenised properties and stress resultant has been developed using *Mathematica* symbolic computation package. Obtained homogenised forces $\left(N_1^0,\ N_2^0,\ Q_1^0\right)$ and moments $\left(M_1^0,\ M_2^0\right)$ were subsequently recalculated to the forces and moments acting directly in each family of ribs using transformation formulae:

$$
\begin{aligned}
N_{1,2}^* &= \left[\left(K_3 N_1 c^2 + K_4 N_2 s^2\right)K^0 \pm SK\left(\sin 2\varphi\right)^{-1}\right]EF,\\
N_3^* &= \left[\left(2Kc^4 + K_4\right)N_2 - 2KN_1 s^2 c^2\right]K^0 E_3 F_3,\\
N_4^* &= \left[\left(2Ks^4 + K_3\right)N_1 - 2KN_2 s^2 c^2\right]K^0 E_4 F_4.
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
M_{1,2}^* &= \{\left[\left(I_3 c^3 + 2Cs^2c^2\right)M_1 + \left(I_4 s^2 + 2Cs^2c^2\right)M_2\right]I_0 \mp H_1(I\sin 2\varphi +\\
&\quad +2Cc^2\,\mathrm{ctg}\,2\varphi + C_4 \sin^{-1} 2\varphi)^{-1}\}EJ_1,\\
M_3^* &= \left[\left(2Ic^4 + I_4 + 2Cs^2c^2\right)M_2 - 2M_1 s^2c^2\left(I - C\right)\right]I_0 E_3 J_{13},\\
M_4^* &= \left[\left(2Is^4 + I_3 + 2Cs^2c^2\right)M_1 - 2M_2 s^2c^2\left(I - C\right)\right]I_0 E_4 J_{14},\\
H_{1,2}^* &= \left\{\pm\left[\left(Is^2 + I_3\right)M_1 - \left(2Ic^2 + I_4\right)M_2\right]I_0 + 2H_1\beta_{31}^{-1}\cos 2\varphi\right\}GJ_3,\\
H_3^* &= -2G_3 J_{33} H_1 \beta_{31}^{-1}, \qquad H_4^* = -2G_4 J_{34} H_1 \beta_{31}^{-1},\\
I_0 &= \left[2I\left(I_3 c^4 + I_4 s^4\right) + I_3 I_4 + 2Cs^2c^2\left(2I + I_3 + I_4\right)\right]^{-1}.
\end{aligned}
\tag{9}
$$

where $K^0 = \left[2K\left(K_3 c^4 + K_4 s^4\right) + K_3 K_4\right]^{-1}$. In the first of the above formulae a negative sign before the second summand is used for the ribs of the second family.

FEM model was created using commercially available FEM code *MSC/NASTRAN*. The linear elastic static analysis has been performed.

Resultant stresses acting in the ribs that belong to the first, second and third family respectively and also the total z displacement are plotted (Figure 4-Figure 6) versus the height of the cylinder (coordinate z).

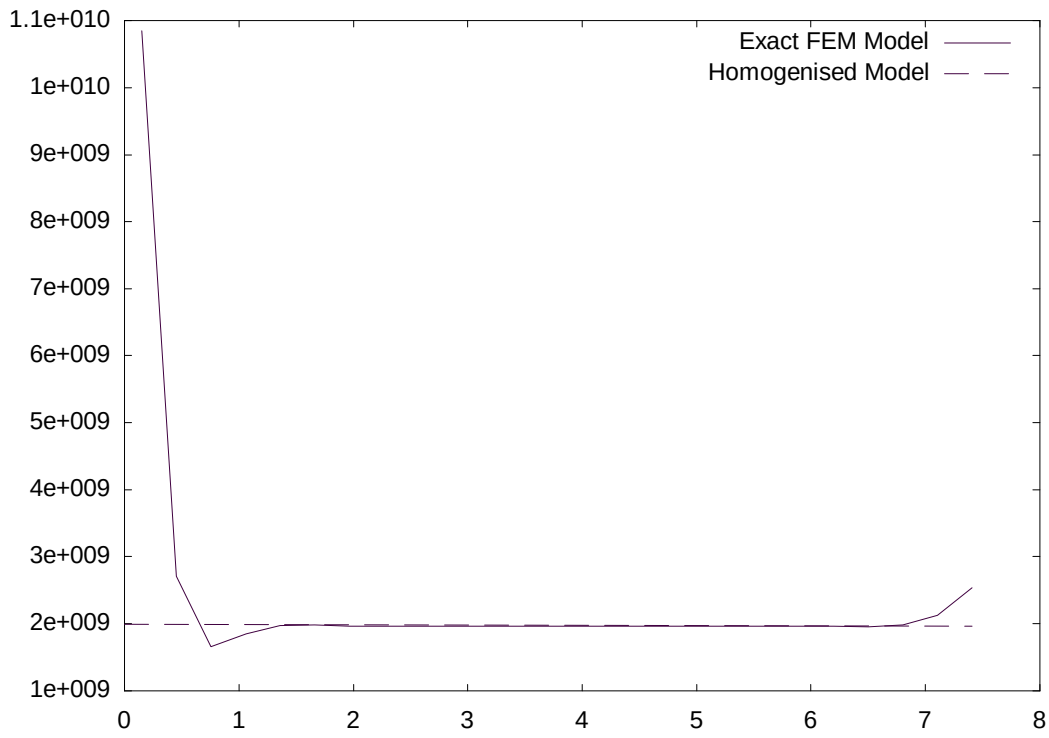

Figure 4 Axial Stress in the vertical members

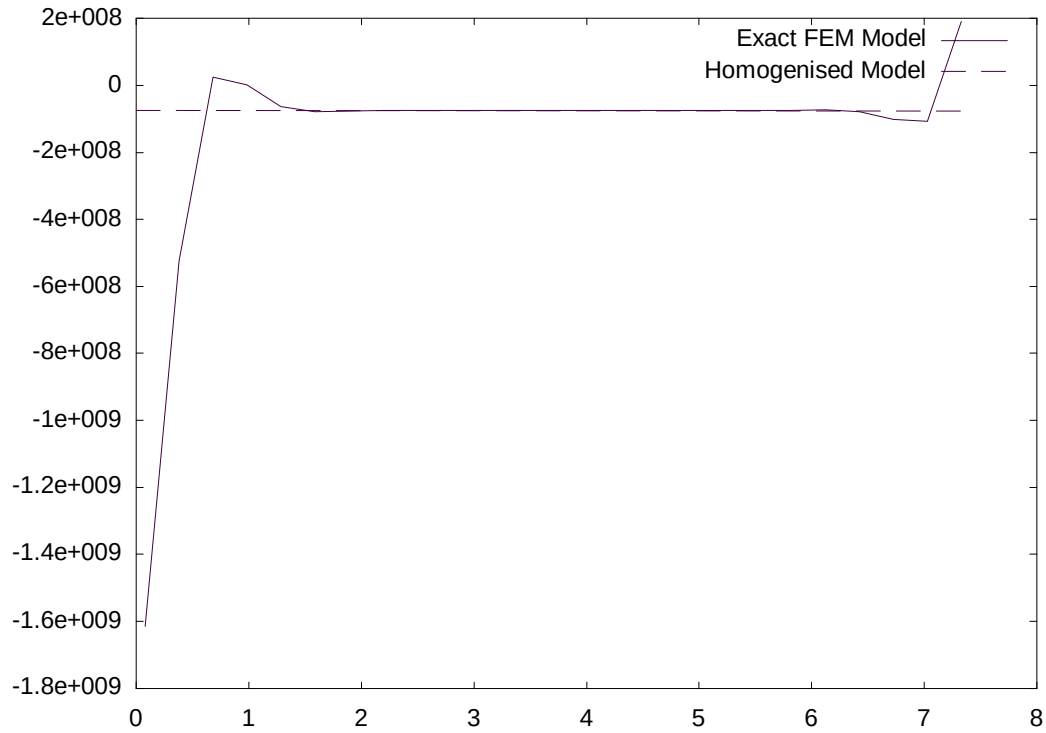

Figure 5 Axial Stress in the diagonal members

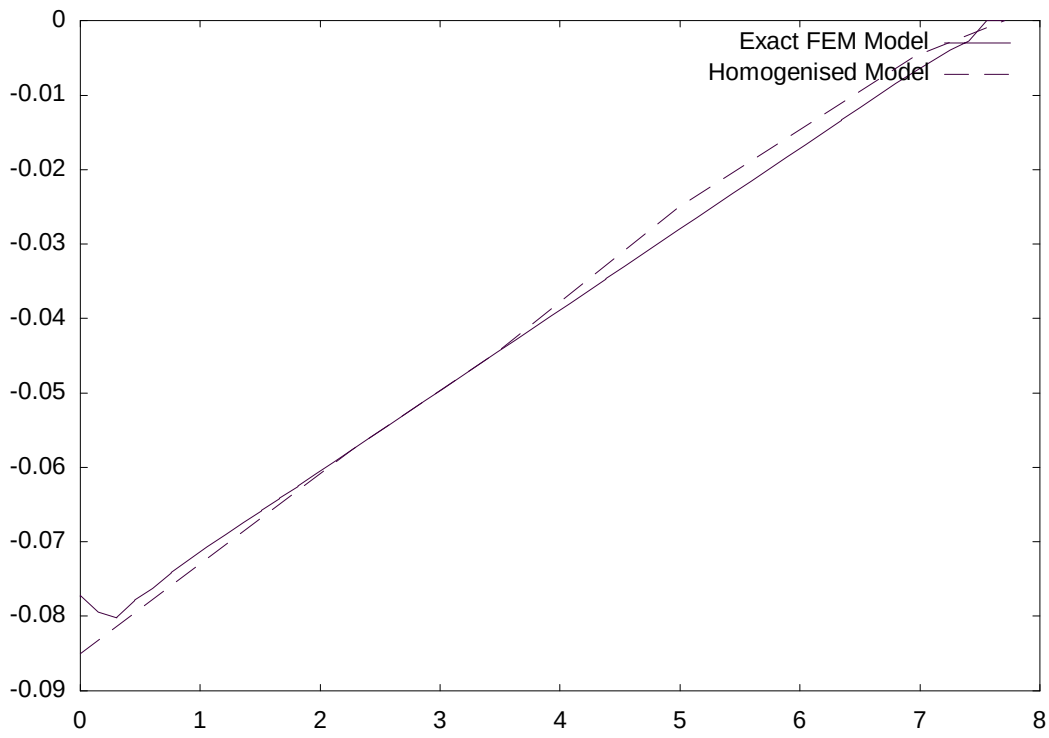

Figure 6 Z Displacement

As it seen the results (Figure 4-Figure 6) show good correlation within the tolerance of 10% for most of the elements of the model except for those elements subjected to the local effects. The considerable difference of the results can be seen in the vicinity of the constrained contour and the loaded edge of the cylinder. This inconsistency can be overcome by taking into account boundary effects which are not discussed in the present paper.

On the basis of derived system of differential equations (2) the variety of optimisation tasks can now be performed:

1. Minimisation of the total weight of the structure subjected to the variable design parameters (cell configuration, cross-section of the ribs, etc.).
2. Maximisation of the applied load subjected to the same set of design variables.

The first case is obviously more complicated because the objective function (weight) is non-linear and optimisation requires constant verification of another non-linear function (strength ratio of the structural members). The practical realisation of such optimisation task requires incorporation of complex non-linear programming algorithms.

For the second case the optimisation task will be much simplified since it deals with only one function. This function is the objective function that is derived in analytical form from the solution of boundary value problem (2). The failure coefficients enter the expression of the considered objective function and their calculation performed inside the objective function during optimisation.

CONCLUSIONS

The verification of the ESM has been undertaken by means of comparing it with the exact FEM model. The general verification of the homogenisation concept has been made and the estimation of the accuracy of the results has been performed. The comparison of the stresses and displacements of the elements in two models shows that the difference in the results obtained from the analysis of exact and equivalent stiffness models does not exceed 10% which is acceptable. The inconsistency in stress resultants is localised in the boundary areas where the load and constraints are applied. At this stage the developed homogenised model fails to predict stress resultants with the reasonable accuracy in the vicinity of the areas where boundary effects are taking place. The homogenisation approach shows that complex and extensive stress analysis of exact FEM model can be substituted with analysis on the basis of equivalent stiffness model within very reasonable range of accuracy.

The advantage of the proposed approach is that it allows to make quick changes in the design of the structure. These changes could be quite complex and laborious task when performing analysis on the basis of FEM model. The later requires recreation and remeshing of the whole structure. In the case of homogenised model all geometric changes can be made by changing the system of equations (2). Moreover, any geometric or other properties can be used as an optimisation variable which is not always possible in the case of utilisation of commercially available FEM packages.

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