OPTIMAL PIEZO-PATCH DESIGN OF COMPOSITE BEAMS UNDER UNCERTAIN LOADS FOR MINMAX DEFLECTION

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SUMMARY: Shape of a laminated beam is controlled by an optimally placed piezo actuator so as to minimize its maximum deflection. The locations and magnitudes of the external loads are not known *a priori* and belong to a specified load uncertainty domain. The optimal actuator location is obtained for any load combinations and locations which are determined so as to produce the maximum deflection corresponding to the worst case of loading. In this sense, loading uncertainties lead to an anti-optimization problem which is coupled to the optimization problem via the design parameter and loading. Numerical results are given to assess the effect of load uncertainty and actuator length on the actuator location and the design efficiency. It is shown that the controlled beams can have less than half the maximum deflection of uncontrolled beams

KEYWORDS: Smart structures, piezoelectric materials, design under uncertainty, composite beams, optimal design.

INTRODUCTION

Load uncertainties occur quite often in practice as the precise locations and magnitudes of loads acting on a structure may not be known *a priori*. In such cases, the location of a piezo actuator has to be selected on the basis of the worst-case loading. Even though uncertainty in loading is a fairly common occurance, shape control has been mostly studied for structures under deterministic loads. Studies on the deflection control using piezoelectric actuators include [1] for plates, and [2] - [6] for beams. Placement of actuators in adaptive structures was studied for trusses in [7] where further references on this topic are given. Optimal placement of piezoelectric actuators were studied for beams in [3], [4], [6], and [8]. In the above studies, the loading conditions were taken as deterministic. Closed loop shape control formulation was given in [9] for composite beams for the case when the external loads are not known precisely.

Examples of design optimization of structural elements under uncertain loading conditions were given in [10] - [12]. In these studies the design variables of the problem were ply angles and thicknesses of the composite structures.
Minimization of the maximum deflection of composite beams using an optimally placed piezo actuator is the subject of the present study. The magnitudes and locations of the external loads acting on the beam are not known *a priori*. As such the applied loads belong to an uncertainty domain which determines the maximum allowable values of loads. The loads could act at any point on the beam and in this sense the load locations are also uncertain. The uncertainty variables (load magnitudes and locations) are determined so as to produce the maximum deflection and this constitutes the anti-optimization problem. The optimization problem consists of determining the location of the piezo actuator optimally under any load combinations. The optimal value of the design variable (actuator location) and the anti-optimal values of the uncertainty variables depend on each other leading to a nested anti-optimization/optimization problem. The simultaneous solution of this problem leads to a robust actuator design producing the min-max deflection beam under any loadings within the uncertain load domain.

Numerical results are given for simply supported beams and for beams with an extended section. The efficiency of the design is assessed by comparing the deflections of controlled beams with uncontrolled ones. It is observed that the efficiency increases with increasing actuator length. It also depends on the load combinations and the boundary conditions of the beam.

ANTI-OPTIMIZATION AND OPTIMIZATION PROBLEMS

A uniform beam of length L is under the action of transverse loads $0 < P_i < P_{i,max}$ acting at $\xi_i \in [0, L]$ and moments $M_{j,min} < M_j < M_{j,max}$ acting at $\zeta_j \in [0, L]$ where $i = 1, 2, \ldots, I$, $j = 1,2, \ldots, J$. The set defined by

$$U = \left\{ \left(P_i, \xi_i, M_j \right) \middle| \; 0 \le P_i \le P_{i,max}, \xi_i \in \left[0, L \right], M_{j,min} \le M_j \le M_{j,max} \right\} \tag{1}$$

is the load uncertainty domain. The locations ξ_i of the loads are also unknown and they are to be determined as part of the solution of the anti-optimization problem. The locations ζ_j of concentrated moments M_j are specified as input parameters in the present study.

The magnitudes of the loads P_i and moments M_j can take any value within the uncertainty domain U and the uncertain load locations can take any value in [0, L]. The uncertain quantities P_i, ξ_i, $M_j \in U$ are unknown variables of the anti-optimization problem which consists of determining the pair (P_i, ξ_i) and M_j such that the maximum deflection of the beam is maximized.

Let $w(x; \alpha, \beta)$ denote the mid-plane deflection of the beam for a loading in U where α is the design variable to be defined later and $\beta = (P_i, \xi_i, M_j)$ is the set of uncertainty variables. The maximum deflection of the beam is given by

$$\left\| w(\bullet;\alpha,\beta)\right\| = \max_{0\le x\le L} \left| w(x;\alpha,\beta)\right| \tag{2}$$

The anti-optimization problem can be stated as:

Determine the uncertainty variable β such that

$$\left\| w\left(\bullet;\alpha,\beta^{*}\right)\right\| = \max_{\beta\in U} \left\| w(\bullet;\alpha,\beta)\right\| \tag{3}$$

Thus, the uncertainty variables are determined for each loading case so as to maximize the maximum deflection which constitutes the solution of the anti-optimization problem.

Next the optimization problem and the design variables are defined. A piezo actuator of length a is symmetrically bonded on the top and bottom surfaces of the beam. Its location is defined by b which indicates the distance between the left hand support and the actuator. A voltage V is applied on the piezoactuator the sign of which is given by sgn(V) = V/ | V |. The actuator location b and the sign of the voltage sgn (V) serve as the design variables of the optimization problem so that α = (b, sgn (V)) if $V \neq 0$ and $\alpha = \alpha_u = b_u$ if V = 0 (uncontrolled case). The beam deflection is to be minimized by choosing the location $0 \le b < L-a$ of the actuator and the sgn (V) optimally. Thus the optimization problem can be stated as

$$\left\| w\left(\bullet;\alpha^{*},\beta^{*}\right)\right\| = \min_{0\le b\le \overline{L}} \left\| w\left(\bullet;\alpha,\beta^{*}\right)\right\| \tag{4}$$

where $\overline{L}$ = L-a. Clearly the optimal value of α depends on the anti-optimal values of P_i, ξ_i and M_j and vice versa leading to a nested optimization/anti-optimization problem.

BASIC EQUATIONS

The beam is of rectangular cross-section with height H and width B. The piezo-actuator has width B, length a and thicknesses h_p which are activated by the application of out-of-phase electric potentials with a voltage V. The x-coordinate is taken along the mid-plane of the beam so that |z| < H/2 indicates the beam region and |z| < H/2 the piezo layers which are denoted by S_b and S_p, respectively.

Let M(x) denote the moment at x under the loading U. The equations governing the flexural bending of the beam are given by

$$M(x) = D_1\, w'' \quad \text{for} \quad x \in S_b \tag{5}$$

$$M(x) = D_2\, w'' - M_p \quad \text{for} \quad x \in S_p \tag{6}$$

where D_i are the stiffness constants derived below, M_p is the moment generated by the piezo actuators, and the prime superscript stands for differentiation with respect to x.

Let ε_x denote the axial strain at x and u(x) the displacement in the x-direction for loading $\beta \in U$. Then

$$\varepsilon_x = u' + z\, w'' \tag{7}$$

The expressions for the stiffnesses D_1 and D_2 will be derived for a laminated beam with the thickness of the k-th layer denoted by t_k with the total number of layers including the piezo layers denoted by K. The stress $\sigma_x^{(k)}$ for the k-th layer is given by

$$\sigma_x^{(k)} = E_x^{(k)}\, \varepsilon_x^{(k)} - E_x^{(p)}\, d_{31}\, \phi_3 \tag{8}$$

where

$$E_x^{(k)} = \left(E_{11}^{-1} \cos^4 \theta_k + \left(G_{12}^{-1} - 2\nu_{12}\, E_{11}^{-1}\right)\cos^2 \theta_k \sin^2 \theta_k + E_{22}^{-1} \sin^4 \theta_k\right)^{-1} \tag{9}$$

with E_{11}, E_{22}, G_{12} and ν_{12} denoting the elastic constants of the material of the k-th layer. In Eqn 8, d_{31} is the piezoelectric constant and ϕ_3 is the electric field intensity given by $\phi_3 = V/h_p$. Let h_k denote the signed distance between the mid-plane and top of the k-th layer with $h_o = H/2 + h_p$ and $h_K = -H/2 - h_p$. The stress resultant N_x in the x-direction can be computed from

$$N_x = B \sum_{k=k_o}^{K_o} \int_{h_{k-1}}^{h_k} \sigma_x^{(k)}\, dz \tag{10}$$

using Eqns 7 and 8 where $K_o = K - 1$, $k_o = 2$ for $x \in S_b$ and $K_o = K$, $k_o = 1$ for $x \in S_p$. This computation gives

$$\begin{aligned} N_x &= A_x\, u' + B_x\, w'' && \text{for } x \in S_b \\ N_x &= A_x\, u' + B_x\, w'' - N_x^{(p)} && \text{for } x \in S_p \end{aligned} \tag{11}$$

where

$$\left(A_x, B_x\right) = B \sum_{k=k_o}^{K_o} E_x^{(k)} \left(h_k - h_{k-1},\ \frac{1}{2}\left(h_k^2 - h_{k-1}^2\right)\right) \tag{12}$$

which gives different A_x and B_x values for $x \in S_b$ and for $x \in S_p$ and

$$N_x^{(p)} = B \int_{S_p} E_x^{(p)} \, d_{31} \, \phi_3 \, dz \tag{13}$$

Similarly, the moment resultant M_x is given by

$$M_x = B \sum_{k=k_o}^{K_o} \int_{h_{k-1}}^{h_k} \sigma_x^{(k)} \, z \, dz = B_x \, u' + D_x \, w'' - M_x^{(p)} \tag{14}$$

so that

$$\begin{aligned} M_x &= B_x \, u' + D_x \, w'' && \text{for } x \in S_b \\ M_x &= B_x \, u' + D_x \, w'' - M_x^{(p)} && \text{for } x \in S_p \end{aligned} \tag{15}$$

where

$$D_x = \frac{1}{3} B \sum_{k=k_o}^{K_o} E_x^{(k)} \left(h_k^3 - h_{k-1}^3 \right) \tag{16}$$

$$M_x^{(p)} = B \int_{S_p} E_x^{(p)} \, d_{31} \, \phi_3 \, z \, dz \tag{17}$$

Since there is no force in the axial direction, $N_x = 0$ and from Eqn 11 it follows that

$$u' = -\left(B_x / A_x \right) w'' + N_x^{(p)} / A_x \quad \text{for } 0 \le x \le L \tag{18}$$

where $N_x^{(p)} = 0$ for $x \in S_b$. Substituting u' from Eqn 18 into the moment expressions 15, we obtain

$$M_x = \left(D_x - B_x^2 / A_x \right) w'' + \left(B_x / A_x \right) N_x^{(p)} - M_x^{(p)}, \quad 0 \le x \le L \tag{19}$$

where $N_x^{(p)} = M_x^{(p)} = 0$ for $x \in S_b$. Comparing Eqns 5, 6, and 19, we obtain

$$\begin{aligned} D_1 &= D_x - B_x^2 / A_x && x \notin S_p \\ D_2 &= D_x - B_x^2 / A_x && x \in S_b \cup S_p \\ M_p &= M_x^{(p)} - \left(B_x / A_x \right) N_x^{(p)} && x \notin S_b \end{aligned} \tag{20}$$

METHOD OF SOLUTION

Equations 5 and 6 have to be solved for a given loading and actuator location. These equations are solved by finite differences by dividing each region R_i of the beam into n_i intervals. Thus $R_1 \in [0, b]$, $R_2 = [b, b + a]$ and $R_3 = [b + a, L]$, and $\Delta x_1 = b/n_1$, $\Delta x_2 = a/n_2$, $\Delta x_3 = (L - a - b)/n_3$ where Δx_i is the interval length in region R_i. The finite difference expressions for each interval are given by

$$\delta^2 w_j = M_x(x_j)/D_1 \quad \text{for } x \in S_b \tag{21}$$

$$\delta^2 w_j = \left(M_x(x_j) + M_p\right)/D_2 \quad \text{for } x \in S_p \tag{22}$$

where $\delta^2 w_j = (w_{j+1} - 2w_j + w_{j-1})/\Delta x_i^2$, $j = 1, 2, \ldots, n_1$ and $i = 1$ for $x \in R_1$, $j = n_1 + 1, \ldots, n_1 + n_2$, $i = 2$ for $x \in R_2$, etc. At the end points of the piezo actuator ($x = b$ and $b + a$) an average stiffness given by $D_a = (D_1 + D_2)/2$ is taken. The above formulation leads to a system of linear equations, the solution of which gives the deflection w_j at node j.

The maximum deflection computed by the finite difference method gives the objective of the design problem. In order to minimize the maximum deflection under load uncertainties, the anti-optimization problem 3 and the optimization problem 4 have to be solved. Since these problems are coupled, they have to be solved simultaneously. The design parameter is optimized using a one-dimensional optimization algorithm and in the optimization procedure the usual analysis phase corresponds to the solution of the anti-optimization problems.

NUMERICAL RESULTS

In the following, b_{opt} denotes the optimal b, i.e., the optimal distance between the left hand support and the left end point of the piezo actuator. The effectiveness of piezo control can be studied by defining an efficiency index which gives the percent decrease in the uncontrolled deflection as compared to the controlled one, viz.,

$$I_f = \left(\frac{w_{max}(\alpha_u; 0)}{w_{max}\left(\alpha_{opt}; V\right)} - 1 \right) \times 100 \tag{23}$$

where $\alpha_u = b_u$ which is taken as $b_u = (L - a)/2$, i.e., the actuator is located centrally and $w_{max}(\alpha_u;0) = \left\| w(\bullet;\alpha_u,\beta^*) \right\|$ with $V = 0$ is the maximum deflection of the uncontrolled beam. For the controlled beam $w_{max}(\alpha_{opt};V) = \left\| w(\bullet;\alpha_{opt},\beta^*) \right\|$ with $V > 0$ and $\alpha_{opt} = (b_{opt}, \mathrm{sgn}(V)_{opt})$. Thus the comparison is made with a passive beam where the piezo actuator contributes only to the stiffness of the beam.

The results are presented in dimensionless form by using the length L of the beam to non-dimensionalize various quantities:

$$w_n = w_d / L \ , \ a_n = a_d / L \ , \ b_n = b_d / L \ , \ \xi_n = \xi_d / L \qquad (24)$$

where the subscripts n and d denote non-dimensional and dimensional quantities. In the rest of the discussion, the subscript n is dropped from the notation. The results are given for a composite beam made of graphite-epoxy (T300/5280) for which the elastic constants are E_{11} = 181.0 GPa, E_{22} = 10.30 GPa, G_{12} = 7.17 Gpa, and ν_{12} = 0.28.

The piezo-actuators are made of PZT-5H piezoceramic material with the properties E_p = 62 GPa and $d_{31} = -274 \times 10^{-12}$ m/V (see [13]). In all of the calculations the beam length is taken as L = 1m, the thickness as H = 5 mm, the width as B = 10 mm, the actuator thicknesses as h_p = 1 mm, and the beam stacking sequence is chosen as (0°/90°/90°/0°). Thus the stacking sequence is symmetrical and is given by (PZT/0°/90°/90°/0°/PZT) for $x \in S_p$. The allowable voltage for piezoceramic materials is around 500–1000 volts/mm of piezo thickness (see [13]). In the present study, the voltage is taken as V = 500 volts in all cases.

In the finite difference scheme $\Delta x = 0.01L$ in all regions. The distances b_{opt} are computed as multiples of Δx. Thus the optimal locations are obtained up to an accuracy of 0.01L.

Table 1 shows the values of b_{opt} and I_f for the maximum load P_{max} for a simply supported beam under the actions of a concentrated load $0 < P < P_{max}$ acting at $x = \xi_a$ and moment -10 Nm $< M_r <$ 10 Nm applied at x = L. In this case the anti-optimal loading is given by $P = P_{max}$ and M_r = 10 Nm with the anti-optimal load location ξ_a depending on the length of the actuator and the magnitude of P_{max} where sgn $(V)_{opt}$ = +1. At low values of P_{max}, the actuator is closer to the right hand support as the design is dominated by the moment loading. As P_{max} increases, the actuator moves left, i.e., b_{opt} decreases. Higher actuator length leads to higher design efficiency since 8% < I_f < 38% for a = 0.2 and 18% < I_f < 116% for a = 0.4.

Table 2 shows the same values as in Table 1 with $0 < P < P_{max}$ and -20 Nm $< M_r <$ 10 Nm for a simply supported beam. For this case the anti-optimal loading is given by P = 0 and $M_r = -20$ Nm for low values of P_{max} with the optimal sgn (V) = –1 and by $P = P_{max}$ and M_r = +10 Nm for high values of P_{max}

TABLE 1: *Optimal actuator location and efficiency index for P_{max} for load case $0 < P < P_{max}$, and $|M_r| < 10$ Nm*

	a = 0.3		**a = 0.4**	
P_{max} (N)	**b_{opt}**	**I_f (%)**	**b_{opt}**	**I_f (%)**
0	0.49	37.5	0.41	112.7
10	0.46	25.9	0.37	69.4
20	0.45	19.8	0.36	49.7
30	0.44	16.0	0.35	38.7

40	0.43	13.4	0.34	32.2
50	0.43	11.5	0.33	26.7
60	0.42	10.1	0.33	23.1
70	0.42	9.0	0.33	20.4
80	0.42	8.1	0.32	18.2
90	0.42	7.4	0.32	16.4
100	0.42	6.7	0.32	15.0

with optimal sgn (V) = +1. Thus for low values of P_{max}, the loading M_r = –20 Nm dominates the design and the actuator is closer to the right hand support. It is noted that in both cases of loadings ($|M_r|$ < 10 Nm, Table 1 and the present one), the optimal actuator location is sensitive to P_{max} in certain ranges. The design efficiency is not as high as it was in the previous case due to M_r having a larger range.

TABLE 2: *Optimal actuator location and efficiency index for P_{max} for load case $0 < P < P_{max}$, and $-20 < M_r < 10$ Nm*

	a = 0.2		**a = 0.4**	
P_{max} (N)	**b_{opt}**	**I_f (%)**	**b_{opt}**	**I_f (%)**
0	0.50	18.3	0.40	41.0
10	0.50	18.3	0.40	41.0
20	0.50	18.3	0.40	41.0
30	0.50	18.3	0.40	41.0
32	0.47	17.5	0.38	40.2
34	0.44	15.1	0.35	36.9
36	0.43	14.3	0.34	34.1
38	0.43	13.8	0.34	32.8
40	0.43	13.4	0.34	31.6
50	0.43	11.5	0.33	26.7
60	0.42	10.1	0.33	23.1
70	0.42	9.0	0.33	20.4
80	0.42	8.1	0.32	18.2
90	0.42	7.4	0.32	16.4
100	0.42	6.7	0.32	15.0

Next a beam with a hanging section is studied. The beam is simply supported at the left hand support and the other hinge support is located at x = 0.8 L. Table 3 shows the b_{opt} and I_f values for different P_{max} with $0 < P < P_{max}$ acting at $x = \xi_a$ with a moment $|M_l| < 10$ Nm applied at the left hand support x = 0. In this case the anti-optimal loadings are given by M_l = +10 Nm and $P = P_{max}$ ($0 < \xi_a < 0.8$ L) for low values of P_{max} ($\mathrm{sgn}(V)_{opt}$ = +1) and by M_l = –10 Nm and $P = P_{max}$ at ξ_a = L for higher values of P_{max} (sgn $(V)_{opt}$ = –1). As P_{max} increases, the actuator approaches the right support at x = 0.8 L. The efficiency drops substantially as P_{max} increases, but the decrease tapers off after a certain P_{max}.

TABLE 3: *Optimal actuator location and efficiency index for P_{max} for a beam with a hanging section with $0 < P < P_{max}$, and $|M_l| < 10$ Nm*

	a = 0.2		**a = 0.4**	
P_{max} (N)	**b_{opt}**	**I_f (%)**	**b_{opt}**	**I_f (%)**
0	0.22	51.1	0.12	180.1
5	0.23	43.0	0.16	121.9
10	0.24	37.1	0.20	85.9
15	0.28	29.1	0.22	64.9
20	0.31	21.3	0.24	49.4
25	0.33	15.6	0.26	44.9
30	0.35	14.1	0.27	42.0
40	0.38	13.4	0.29	37.4
50	0.41	13.3	0.31	35.3
60	0.43	13.2	0.32	32.9
70	0.44	12.8	0.33	31.5
80	0.46	13.1	0.34	30.8
90	0.47	13.0	0.35	29.8
100	0.48	13.1	0.35	29.0

CONCLUSIONS

The problem of locating a piezo actuator optimally on a beam under uncertain loads was studied. The locations and magnitudes of the loads are not known *a priori* and have to be determined to produce the least favorable loading condition with respect to the maximum deflection. The objective of the design is the minimization of the maximum deflection under any load combination and this is achieved by controlling the deflection using a piezo actuator of a given length.

The loads acting on the beam are defined in an uncertainty domain and maximizing the deflection over the loads within this domain constitutes the anti-optimization problem. The optimization and anti-optimization problems are coupled through the design variable (actuator location) and the loading. The solution procedure requires the solution of the anti-optimization problem for a given value of the design variable which is optimized iteratively.

Results were given for beams with different boundary conditions and uncertainty domains. The effectiveness of the piezo control is assessed by comparing the maximum deflections of controlled and uncontrolled beams. It is observed that the design efficiency depends largely on the uncertainty domain, i.e., the load configuration and the maximum values of the load magnitudes have considerable effect on design efficiencies which could be more than 100%.

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