THE SMOOTH INDENTATION OF A CYLINDRICAL INDENTOR AND ANGLE-PLY LAMINATES

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SUMMARY: The indentation between cylindrical indentor and angle-ply composites is investigated. An exact Green‘s function for the surface displacement of an angle-ply beam is derived using Jones’s displacement formulation. By matching the displacements of the top surface of the angle-ply laminate and those of the indentor within the contact region, we can find the contact stress distribution, the magnitude of contact force and the indentation.

In order to speed up the formulation of fundamental deflection mode shape of Green’s function, method of initial function (MIF) is adopted to find the displacements in the top and bottom surface of an angle-ply laminate. In the MIF, the order of system equation is only 3×3, which will reduce the computation time.

Numerical results for angle-ply laminates subjected to cylindrical indentation are evaluated to verify the applicability of the proposed method.

KEYWORDS: angle-ply laminates, indentation, point matching method, method of initial functions

INTRODUCTION

The contact problems of composite laminate are important in the analysis of foreign object impact and flexural measurement of composite materials. Several authors have studied the contact behaviors between a cylindrical indentor and orthotropic beams[1,2,3]. Sankar solved the

contact problem of orthotropic beam by the superposition of half space solution and beam solution[1]. By applying the exact displacement solution for cylindrical bending of orthotropic beams, the exact Green's function is formulated in solving the contact problems[3]. However the indentation between cylindrical indentor and angle-ply composites is seldom investigated. In this study, an exact Green's function for the surface displacement of an angle-ply beam is derived using Jones's[4] displacement formulation. This Green's function satisfies the displacement and stress continuity condition at the interface, and the prescribed stress condition at the top and bottom surfaces. By matching the displacements of the top surface of the angle-ply laminate and those of the indentor within the contact region, we can find the contact stress distribution, the magnitude of contact force and the indentation.

In constructing the exact Green's function, we have to solve a matrix of order 6N×6N, with M times, where N is the ply number and M is the number of basic flexural modes. If the ply number increases, the computation time will increase drastically. In order to speed up the calculation, method of initial function (MIF)[5] is adopted to find the displacements in the top and bottom surface of an angle-ply laminate. In the MIF, the order of system equation is only 3×3, which will reduce the computation time.

Numerical examples are illustrated to demonstrate the contact behaviors of angle-ply laminates.

CYLINDRICAL BENDING OF ANGLE-PLY LAMINATES

In order to study the contact behavior of angle-ply laminates, the cylindrical bending problems of an anisotropic laminate is investigated first which will provide the Green's function for contact problem. Assuming plane stain condition in the x-direction, an anisotropic beam has width b and thickness h in the z-direction is shown in Figure 1. The strain field is defined as

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0,\ \ \varepsilon_y = \frac{\partial \mathrm{v}}{\partial y}\ ,\varepsilon_z = \frac{\partial \mathrm{w}}{\partial z}$$

$$\gamma_{yz} = \frac{\partial \mathrm{v}}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{xz} = \frac{\partial \mathrm{u}}{\partial z},\quad \gamma_{xy} = \frac{\partial \mathrm{u}}{\partial y} \tag{1}$$

The constitutive relation for a generally anisotropic material is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{2}$$

In which c_{ij} is the elastic constant for each layer. The beam is simply supported along y=0 and y=b, with the following boundary conditions

$$w(0,z)=w(b,z)=0 \tag{3a}$$

$$\sigma_{yy}(0,z)=\sigma_{yy}(b,z)=0 \tag{3b}$$

$$\sigma_{xy}(0,z)=\sigma_{xy}(b,z)=0 \tag{3c}$$

$$\sigma_{zz}(y,0)=\sigma_{xz}(y,0)=\sigma_{yz}(y,0)=0 \tag{3d}$$

$$\sigma_{zz}(y,h)=q(y)\,,\quad \sigma_{xz}(y,h)=\sigma_{yz}(y,h)=0 \tag{3e}$$

If the inertia force is neglected, the equilibrium equation can be written as

$$\sigma_{xy,y}+\sigma_{xz,z}=0 \tag{4a}$$

$$\sigma_{yy,y}+\sigma_{yz,z}=0 \tag{4b}$$

$$\sigma_{yz,y}+\sigma_{zz,z}=0 \tag{4c}$$

The Displacement fields for this problem can be expressed as

$$\begin{aligned} u(y,z)&=\sum_{n=1}^{\infty}U_n(z)\cos\frac{n\pi y}{b}\\ v(y,z)&=\sum_{n=1}^{\infty}V_n(z)\cos\frac{n\pi y}{b}\\ w(y,z)&=\sum_{n=1}^{\infty}W_n(z)\sin\frac{n\pi y}{b}\end{aligned} \tag{5}$$

The above candidates satisfy Eqns. (3a), (3b) and (3c). When substitute Eqn. (5) into Eqn.(4) and Eqn.(2), the following equations can be derived.

$$-p^2c_{66}\,U(z)+c_{55}U''(z)-p^2c_{26}\,V(z)+c_{45}V''(z)+p(c_{36}+c_{45})W'=0 \tag{6}$$

$$-p^2c_{26}\,U(z)+c_{45}U''(z)-p^2c_{22}\,V(z)+c_{44}V''(z)+p(c_{23}+c_{44})W'=0 \tag{7}$$

$$-p(c_{45}+c_{36})\,U'(z)-p(c_{44}+c_{23})\,V'(z)+c_{33}W''(z)-p^2c_{44}\,W(z)=0 \tag{8}$$

In which p=$n\pi/b$, the subscript n of U, V and W are omitted for simplicity. In order to solve the $U_i(z), V_i(z)$ and $W_i(z)$, we let [4]

$$\begin{aligned} U_i(z)&=U_{i0}e^{\lambda z}\\ V_i(z)&=V_{i0}e^{\lambda z}\\ W_i(z)&=W_{i0}e^{\lambda z}\end{aligned} \tag{9}$$

$$\begin{aligned} &a_1=c_{55}\,,\quad a_2=c_{44}\,,\quad a_3=c_{33}\\ &b_1=-p^2c_{66}\,,\quad b_2=-p^2c_{22}\,,\quad b_3=-p^2c_{44}\\ &c_1=c_{45}\,,\quad d_1=p^2c_{26}\,,\quad e_1=p(c_{36}+c_{45})\quad ,e_2=p(c_{23}+c_{44})\end{aligned} \tag{10}$$

By substituting Eqns. (9) and (10) into Eqns. (6)-(8), the nontrivial solutions exist if the following relation holds

$$\det\begin{vmatrix}\lambda^2a_1+b_1 & \lambda^2c_1-d_1 & \lambda e_1\\ \lambda^2c_1-d_1 & \lambda^2a_2+b_{22} & \lambda e_2\\ -\lambda e_1 & -\lambda e_2 & \lambda^2a_3+b_3\end{vmatrix}=0 \tag{11}$$

After solving the 6th order characteristic equation of λ , the final form of $U_n(z), V_n(z)$ and $W_n(z)$ can be yielded as

$$U_n(z) = \sum_{i=1}^{6} U_i^* \alpha_{in} e^{\lambda_{in} z}$$
$$V_n(z) = \sum_{i=1}^{6} V_i^* \alpha_{in} e^{\lambda_{in} z} \qquad (12)$$
$$W_n(z) = \sum_{i=1}^{6} W_i^* \alpha_{in} e^{\lambda_{in} z}$$

In which U_i^* , V_i^* and W_i^* are the corresponding eigenvectors when solving Eqn.(11). The only unknowns are α_{in}. For a fixed p, there are six α_{in} ‘s in each layer. These constants can be solved from the interface continuity and the traction conditions at the top and bottom surfaces. If the laminate is made of N layers, the continuity of displacements and stresses at the interface requires that

$$u^k(y,h_k) = u^{k+1}(y,0)$$
$$\mathrm{v}^k(y,h_k) = \mathrm{v}^{k+1}(y,0)$$
$$w^k(y,h_k) = w^{k+1}(y,0)$$
$$\sigma_{zz}^k(y,h_k) = \sigma_{zz}^{k+1}(y,0) \qquad (13)$$
$$\sigma_{yz}^k(y,h_k) = \sigma_{yz}^{k+1}(y,0)$$
$$\sigma_{xz}^k(y,h_k) = \sigma_{xz}^{k+1}(y,0), \qquad k = 1,2,\ldots., N-1$$

In which h_k is the thickness of the k-th layer. The traction conditions on the top and bottom surface impose that

$$\sigma_{zz}^1(y,0) = \sigma_{yz}^1(y,0) = \sigma_{xz}^1(y,0) = \sigma_{yz}^N(y,h_N) = \sigma_{xz}^N(y,h_N) = 0$$
$$\sigma_{zz}^N(y,h_N) = q(y) \qquad (14)$$

If the normal stress on the top surface is

$$q(y) = \sigma_o \sin py \qquad (15)$$

then the coefficients α_{in} can be solved through a set of matrix of order 6N×6N. The vertical displacements w(y,h) and w(y,0) are of great importance in the contact problem.

CYLINDRICAL INDENTATION OF SIMPLY SUPPORTED ANGLE-PLY LAMINATES

Consider a simply supported angle-ply laminate subjected to the indentation of a rigid cylinder as shown in Fig. 2. The radius of the indentor is R. The friction between the indentor and laminate is neglected. The indentation is assumed to be symmetric about the midspan of the beam. The contact length is 2c. In the contact problems the distribution of contact pressure has to be determined. The point matching method is adopted to find the contact stress distributions.

From previous section, the displacement at the top surface of the beam due to a normal stress $\sigma_{zz} = \sigma_o \sin(n\pi y / b)$ applied at z=h is shown to be

$$w(y,h) = W_n(h) \sin \frac{n\pi y}{b} \tag{16}$$

For any arbitrary normal stress q(y) at the top surface, the displacement of the beam will be

$$w(y,h) = \sum_{n=1}^{\infty} q_n W_n(h) \sin \frac{n\pi y}{b} \tag{17}$$

In which

$$\begin{aligned} q_n &= \frac{2}{b}\int_0^b q(y) \sin \frac{n\pi y}{b} dy \\ &= \frac{2}{b}\int_{\frac{b}{2}-c}^{\frac{b}{2}+c} q(y) \sin \frac{n\pi y}{b} dy \end{aligned} \tag{18}$$

From Eqns. (17) and (18), we have

$$\begin{aligned} w(y,h) &= \sum_{n=1}^{\infty} W_n(h) \frac{2}{b} \int_\Omega q(\xi) \sin \frac{n\pi \xi}{b} d\xi \sin \frac{n\pi y}{b} \\ &= \int_\Omega G(y|\xi) q(\xi) d\xi \end{aligned} \tag{19}$$

Where Ω is contact region, and the Green's function is defined as[3]

$$G(y|\xi) = \sum_{n=1}^{\infty} W_n(h) \frac{2}{b} \sin \frac{n\pi y}{b} \sin \frac{n\pi \xi}{b} \tag{20}$$

The relative vertical displacements of any point w(y,h) inside the contact region and the center point w(0.5b,h) can be expressed as[2]

$$\begin{aligned} w(y,h) - w(0.5b,h) &= R\left(1 - \sqrt{1 - \left(\frac{y-0.5b}{R}\right)^2}\right) \approx \frac{(y-0.5b)^2}{2R} \\ &= \int_\Omega [G(y|\xi) - G(0.5b|\xi)] q(\xi) d\xi \end{aligned} \tag{21}$$

The contact region is discretized into several segments, within each segment the contact stress is assumed to be constant. By the least square method and point matching method[1,3], we can find the contact stress distributions.

METHOD OF INITIAL FUNCTIONS

In constructing the Green's function for the contact problems, the vertical deflection at top surface $W_n(h)$ due to a sinusoidal stress is needed. If the layer number increases, it will be time consuming to find the basic deflection mode. The method of initial function[5] is applied to simplify the calculation of $W_n(h)$. From the strain definitions, the stress-strain relation and equilibrium equations, the following equation can be derived[5]:

$$\frac{\partial}{\partial z}\begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix} \tag{22}$$

In which

$$A = \begin{bmatrix} c_{44}/d_1 & -c_{45}/d_1 & 0 \\ -c_{45}/d_1 & c_{55}/d_1 & -\alpha \\ 0 & -\alpha & 0 \end{bmatrix} \tag{23a}$$

$$B = \begin{bmatrix} -c_{66}^{*}\alpha^2 & -c_{26}^{*}\alpha^2 & -\dfrac{c_{36}}{c_{33}}\alpha \\ -c_{26}^{*}\alpha^2 & -c_{22}^{*}\alpha^2 & -\dfrac{c_{23}}{c_{33}}\alpha \\ -\dfrac{c_{36}}{c_{33}}\alpha & -\dfrac{c_{23}}{c_{33}}\alpha & 1/c_{33} \end{bmatrix} \tag{23b}$$

$$d_1 = c_{44}c_{55} - c_{45}^2, \alpha = \frac{\partial}{\partial y}, \alpha^2 = \frac{\partial^2}{\partial y^2},$$

$$c_{ij}^{*} = c_{ij} - \frac{c_{i3}c_{j3}}{c_{33}}, \quad i, j = 1,2,3,6 \tag{24}$$

$$X = \sigma_{xz}, \quad Y = \sigma_{yz}, \quad Z = \sigma_{zz}$$

The material constants for the i-th layer should be substituted in Equations (22) and (23). The solution of Eqn. (22) is shown to be

$$\mathbf{R}(z) = \mathrm{e}^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} z}\mathbf{R}(0) \tag{25}$$

Where

$$\mathbf{R}(z)^T = \{U(z), V(z), Z(z), X(z), Y(z), W(z)\} \tag{26}$$

The state vector **R**(z) in the i-th layer at arbitrary height z can be expressed in terms of the state vector at the initial reference height of that layer. By a successive multiplication of the state vector at each layer, $W_n(h)$ can be found by solving a set of 6×6 simultaneous equation.

NUMERICAL EXAMPLES

A $[\pm 45]_{2S}$ composite laminate made of Hexcel graphite/epoxy is analyzed as an illustrative example. The beam has a width of 10 cm and thickness of 1 cm. Each lamina assumes the following properties,

$$E_1 = 105.71GPa, \quad E_2 = E_3 = 8.57GPa, \quad \upsilon_{12} = \upsilon_{13} = 0.327, \quad \upsilon_{23} = 0.306$$

$$G_{12} = G_{13} = 4.39GPa, \quad G_{23} = 3.05GPa$$

The radius R of the rigid indentor is 7.62cm. The through-the-thickness stress distribution is calculated for contact length 2c=2mm . Figure 3 shows the thickness-wise distribution of the normal stress σ_{zz} . The results at three different positions (the contact center, the contact edge and end of beam) are compared. It seems that the normal stress will have maximum value at the top surface of the contact center. The interlaminar stress distributions of σ_{yz} and σ_{xz} are demonstrated in Figs. 4 and 5. The solutions satisfy the traction free conditions at both top and bottom surfaces. Orthotropic solutions are also compared, in which the effective elastic constants of this angle-ply laminate is calculated using a 3-D effective moduli theory[6]. The orthotropic results are comparable to the layer-wise solutions. Figure 6 shows the in-plane stress distribution of σ_{xx}. The relation of indentation α and contact force is evaluated in Fig. 7. The indentation is defined as the difference of vertical displacement of w(0.5b, h) and w(0.5b, 0). Figure 8 is the contact pressure distribution for different contact length. In order to verify the applicability of method of initial function, results of $W_n(h)$ for different n are revealed in Fig. 9. It seems that both methods agree well for this case. In some other case, if n is large, the successive multiplication of state vectors will cause some numerical instability. If this instability can be overcome, the method of initial function will be an efficient tool for contact analysis.

CONCLUSIONS

A method for the analysis of indentation of angle-ply laminates subjected to rigid cylindrical indentor is proposed. Numerical results show that this method satisfies the given boundary conditions and traction conditions. The contact pressure distribution, magnitude of contact force and indentation can be evaluated. This analytical results will compare to the experimental results in the near future.

REFERENCES

1. Sankar, B. V., “Smooth Indentation of Orthotropic Beams,” Composites Science and Technology, **34**, (1989) 95-111.
2. Sun, C. T. and Sankar, B. V., “Smooth Indentation of an initially stressed orthotropic beam,” Int. J. Solids and Structures, **21**, (1985), 161-176.
3. Wu, E., Chao, J. C. and Yen, C. S., “Smooth Contact of Orthotropic Laminates by Rigid Cylinders,” AIAA Journal, **31**, 10, (1993), 1916-1921.
4. Jones, A. T., “Exact Natural Frequencies and Modal Functions for a Thick Off-Axis Lamina,” J. of Composite Materials, **5**, Oct. (1971), 504-520.
5. Sundara Raja Iyengar, K. T. and Pandya, S. K., “Analysis of Orthotropic Rectangular Thick Plates,” Fibre Science and Technology, **18** (1983), 19-36.
6. Sun, C. T. and Lee, S., “Three-dimensional Effective Elastic Constants for Thick Laminates,” J. of Composite Materials, 1988, **22**, 629-636.