CLOSED-FORM ANALYSIS OF AN ELLIPTICALLY REINFORCED CIRCULAR HOLE IN A LAMINATE PLATE UNDER BENDING

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SUMMARY: The problem of an elliptical doubler reinforcement of a circular hole in an anisotropic plate or laminate under bending is investigated in terms of linear elasticity. The size of the elliptical hole reinforcement is optional and arbitrary anisotropic elastic bending stiffnesses are assumed for the base plate and the reinforcement. The assessment of the respective field quantities like bending moments and normal deflection for the basic laminate and reinforcement domain is performed by complex potential method. This requires an appropriate selection of the complex potentials given by some corresponding series representation inside and outside the reinforcement region. The derived analytical solution enables an effective and accurate assessment of the hole reinforcement.

KEYWORDS: Circular hole, complex potential, analytical solution, hole reinforcement, doubler, laminate, anisotropy, plate problem, bending

INTRODUCTION

The application of laminate light weight constructions is of particular interest in many engineering areas like aerospace construction and automotive industry. For various reasons cutouts like holes are often inevitable e.g. due to the assembly of different substructures by screwed joints or the employment of wiring connections. The presence of such holes leads to a local stress concentration resulting in a possible premature failure of the structure. Thus the investigation of the corresponding field quantities in the hole domain is a serious concern during the design and construction process. A lot of literature is available for the problem of a circular and elliptical hole in an infinitely extended isotropic as well as anisotropic plate under bending [1], [2], [3]. So, the behaviour of an unreinforced hole is well-known. In order to avoid a premature failure of the structure it is of particular interest to reduce the occurring stress concentrations at the hole vicinity. This can be achieved by the application of local hole reinforcements or so-called doublers composed of two patches attached on each side of the plate around the hole domain, (Fig. 1). In this kind of situation, to the authors' knowledge, very little analytical results are available about the reinforcement behaviour. In the present work the doublers are presumed to be of an arbitrary elliptical shape and size aligned concentrically with respect to the circular hole. For the base laminate and the reinforcement arbitrary laminate layups are allowed. The structural analysis of the given problem is performed by complex potential method with appropriately chosen series representations inside and outside the reinforcement region.

COMPLEX POTENTIAL METHOD FOR ANISOTROPIC PLATES UNDER BENDING

Prior to the consideration of the hole reinforcement some basic settings with regard to plate theory and complex method are to be resumed. The investigated plate is supposed to be sufficiently thin so that the behaviour of the composite plate can be described by the classical laminate theory. Thus for the case of a symmetrical layup and a pure bending load the inplane deflections u, v and the transversal deflection w can be

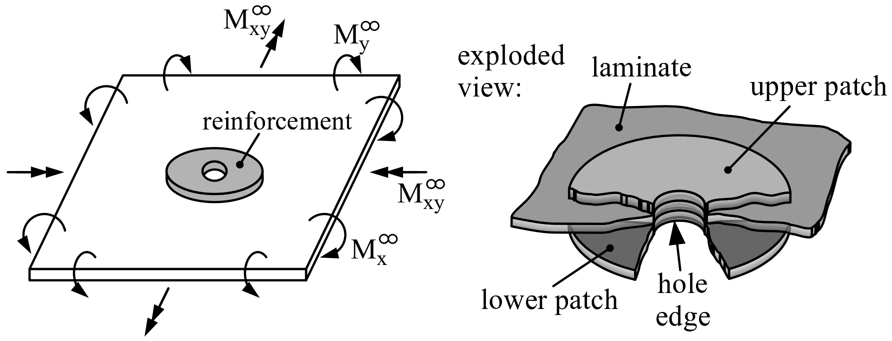

Figure 1: Local hole reinforcement by doublers for a plate under bending

represented by the midplane deflection w_0 and the respective derivatives $w_{0,x}$ and $w_{0,y}$ as follows:

$$\begin{aligned} u(x,y) &= -z\ w_{0,x}(x,y) \quad , \\ v(x,y) &= -z\ w_{0,y}(x,y) \quad , \\ w(x,y) &= w_0(x,y) \quad . \end{aligned} \tag{1}$$

Assuming geometrical linearity the strain components read:

$$\varepsilon_x = u_{,x} \quad , \quad \varepsilon_y = v_{,y} \quad , \gamma_{xy} = u_{,y} + v_{,x} \quad , \tag{2}$$

and can be rewritten as

$$\varepsilon_x = z\,\kappa_x \quad , \quad \varepsilon_y = z\,\kappa_y \quad , \quad \gamma_{xy} = z\,\kappa_{xy} \quad . \tag{3}$$

The midplane curvatures κ_x, κ_y and κ_{xy} result from the midplane deflections w_0:

$$\kappa_x = -w_{0,xx} \quad , \quad \kappa_y = -w_{0,yy} \quad , \quad \kappa_{xy} = -2\,w_{0,xy} \quad . \tag{4}$$

The stated curvatures give the resultant moments by means of Hooke's constitutive law as follows:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad . \tag{5}$$

Herein, the matrix with the components D_{ij} represents the bending stiffnesses of the laminate plate [4]. The resultant moments have to be in accordance with equilibrium in $z-$direction which in the case of vanishing distributed loads means:

$$M_{x,xx} + 2\ M_{xy,xy} + M_{y,yy} = 0 \quad . \tag{6}$$

For a useful representation of the introduced field quantities and for the later analysis two complex potentials $\varphi_1(z_1)$, $\varphi_2(z_2)$ are introduced which are functions of complex variables

$$z_j = x + \mu_j\ y \quad , \quad j = 1,2 \tag{7}$$

where the quantities μ_j are appropriate complex constants. These complex potentials have been suggested first by Muskhelishvili-Kolosov-Lekhnitskii [2], [5], [6] and allow to satisfy the set of basic equations in an identical manner. By means of the complex potentials the field quantities are represented as

$$\begin{aligned} &w(x,y) = 2\,\mathrm{Re}\,[\varphi_j(z_j)] \\ &\kappa_x = -2\,\mathrm{Re}\,[\varphi_j''(z_j)], \quad \kappa_y = -2\,\mathrm{Re}\,[\mu_j^2\ \varphi_j''(z_j)], \quad \kappa_{xy} = -2\,\mathrm{Re}\,[2\ \mu_j\ \varphi_j''(z_j)], \\ &M_x = 2\,\mathrm{Re}\,[d_j\ \varphi_j''(z_j)], \quad M_y = 2\,\mathrm{Re}\,[e_j\ \varphi_j''(z_j)], \quad M_{xy} = 2\,\mathrm{Re}\,[f_j\ \varphi_j''(z_j)], \end{aligned} \tag{8}$$

with appropriately chosen complex constants d_j, e_j, f_j. Within relations (8) prime denotes differentiation with respect to the complex argument. In order to determine the complex constants d_j, e_j, f_j and μ_j the resultant moments and curvatures in equation (5) are substituted by representations (8) which gives:

$$\begin{bmatrix} d_j \\ e_j \\ f_j \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -1 \\ -\mu_j^2 \\ -2\,\mu_j \end{bmatrix} \quad . \tag{9}$$

The substitution of the moments in the equilibrium condition (6) leads to:

$$2\,\mathrm{Re}\left[\left(d_j + 2\,f_j\,\mu_j + e_j\,\mu_j^2\right)\;\varphi_j'''(z_j)\right] = 0 \tag{10}$$

This equilibrium condition has to be satisfied for arbitrarily chosen complex functions. Together with equation (9) and after some rearrangements this leads to a polynomial of degree four for the complex quantities μ_j:

$$D_{22}\,\mu_j^4 + 4\,D_{26}\,\mu_j^3 + (4\,D_{66} + 2\,D_{12})\,\mu_j^2 + 4\,D_{16}\,\mu_j + D_{11} = 0 \quad . \tag{11}$$

The four roots of this equations occur in pairs of complex conjugates μ_1 , μ_2 , $\mu_3 = \overline{\mu}_1$, $\mu_4 = \overline{\mu}_2$. After the complex constants μ_1, μ_2 have been calculated the remaining complex quantities d_j, e_j, f_j can be determined in a straightforward way by means of equation (9).

CLOSED-FORM SOLUTION FOR THE REINFORCED HOLE UNDER BENDING

By means of the given complex representation a closed-form analytical solution is to be considered for the reinforced hole under bending. As shown in Fig. 2 the reinforcement problem is separated into the structural mechanical problem of an infinitely extended plate with an elliptical cutout and the elliptical reinforcement region having a concentrically arranged circular hole. The semi-axes of the elliptical reinforcement shall have the lengths a and b and the hole size is defined by the radius r. Thus the elliptical reinforcement boundary can be described in cartesian coordinates as

$$x^e = a\;\cos\phi \quad , \quad y^e = b\;\sin\phi \;, \tag{12}$$

where the angular parameter ϕ covers the range $0 \le \phi \le 2\,\pi$.
Unless otherwise stated the outer region of the infinite plate is denoted by the index 'I' and the inner reinforcement region by the index 'II'. By virtue of the different stiffness properties the introduced complex quantities d_j, e_j, f_j and μ_j have to be evaluated separately for each region 'I' and 'II'. The outer infinitely extended plate is loaded by the bending moments M_x^∞, M_y^∞ and M_{xy}^∞ at infinity. In addition, the elliptical hole edge is loaded by the Kirchhoff's substitute transversal force $\overline{Q}_n$ and the normal bending moment M_n exerted by the inner reinforcement problem II. The inner subproblem II on the other hand is also loaded by the Kirchhoff's substitute transversal force $\overline{Q}_n$ as well as the normal bending moment M_n correspondingly. Considering each subproblem individually the following series representation is chosen for the respective complex potentials φ_j^{I} in the outer region (subproblem I):

$$\begin{aligned} \varphi_j^{\mathrm{I}\,\prime}(z_j^{\mathrm{I}}) &= A_j\,z_j^{\mathrm{I}} + \sum_{m=1}^{\infty} B_{jm}\,\omega_j^{\mathrm{I}(-m)} \;, \\ \varphi_j^{\mathrm{I}\,\prime\prime}(z_j^{\mathrm{I}}) &= A_j - \sum_{m=1}^{\infty} \frac{m\,B_{jm}}{\omega_j^{\mathrm{I}m}\;\omega_{0j}^{\mathrm{I}}} \;, \end{aligned} \tag{13}$$

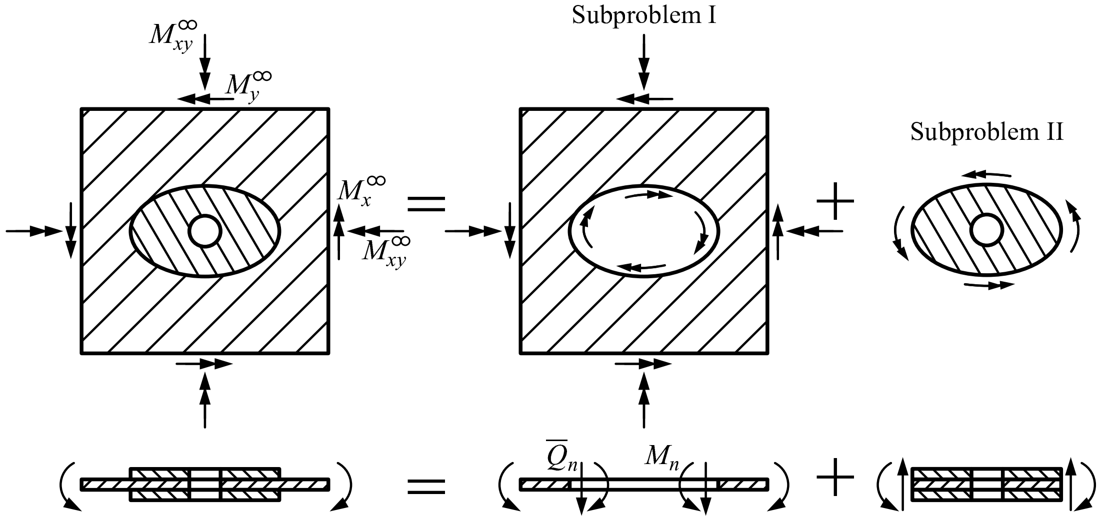

Figure 2: Subproblems of the reinforcement problem

where

$$\omega_j^{\mathrm{I}} = \frac{z_j^{\mathrm{I}} + \omega_{oj}^{\mathrm{I}}}{a - i\,\mu_j^{\mathrm{I}}\,b} \quad , \quad \omega_{oj}^{\mathrm{I}} = \sqrt{z_j^{\mathrm{I}^2} - (a^2 + \mu_j^{\mathrm{I}^2}\,b^2)} \ .$$

This kind of series representation has already been used by Muskhelishvili-Kolosov-Lekhnitskii for the mechanical problem of an elliptical hole in an infinitely extended plate and the assessment of the respective field quantities. The introduced complex quantities A_j and B_{jm} still have to be determined. The complex function ω_j^{I} is a conformal mapping of the outer elliptical region onto an outer unit circle region. Along the elliptical boundary it specializes to the well-known Euler function:

$$\omega_j^{\mathrm{I}(-\mathrm{m})}|_{ellipse} = e^{-m\,i\,\phi} = \cos(m\,\phi) - i\ \sin(m\,\phi) \quad . \tag{14}$$

For subproblem II on the other hand an appropriate superposition of the complex potential representations for an inner ellipse and a circular hole are chosen. Again the structure of the introduced complex potentials for the circular hole is in accordance with the complex representations of Muskhelishvili-Kolosov-Lekhnitskii. For the inner ellipse potentials in terms of series with Faber's polynomials [7] are chosen. Finally the complex potentials φ_j^{II} for the reinforcement are represented in the form

$$\begin{aligned}
\varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}}) &= \underbrace{\sum_{m=1}^{\infty} C_{jm}\,\omega_j^{\mathrm{II}(-m)}}_{\text{circular hole}} + \underbrace{D_{j0} + D_{j1}\,z_j^{\mathrm{II}} + \sum_{m=2}^{\infty} D_{jm}\left(\lambda_j^{\mathrm{II}^m} + t_j^m\,\lambda_j^{\mathrm{II}(-m)}\right)}_{\text{inner ellipse}}, \\
\varphi_j^{\mathrm{II}\prime\prime}(z_j^{\mathrm{II}}) &= \underbrace{-\sum_{m=1}^{\infty} \frac{m\,C_{jm}}{\omega_j^{\mathrm{II}^m}\ \omega_{0j}^{\mathrm{II}}}}_{\text{circular hole}} + \underbrace{D_{j1} + \sum_{m=2}^{\infty} \frac{m\,D_{jm}}{\lambda_{0j}^{\mathrm{II}}}\left(\lambda_j^{\mathrm{II}^m} - t_j^m\,\lambda_j^{\mathrm{II}(-m)}\right)}_{\text{inner ellipse}} \ ,
\end{aligned} \tag{15}$$

where

$$\omega_j^{\mathrm{II}} = \frac{z_j^{\mathrm{II}} + \omega_{0j}^{\mathrm{II}}}{r\,(1 - i\,\mu_j^{\mathrm{II}})} \quad , \quad \omega_{oj}^{\mathrm{II}} = \sqrt{z_j^{\mathrm{II}^2} - r^2\,(1 + \mu_j^{\mathrm{II}^2})} \ ,$$

$$\lambda_j^{\mathrm{II}} = \frac{z_j^{\mathrm{II}} + \lambda_{oj}^{\mathrm{II}}}{a - i\,\mu_j^{\mathrm{II}}\,b} \quad , \quad \lambda_{oj}^{\mathrm{II}} = \sqrt{z_j^{\mathrm{II}^2} - (a^2 + \mu_j^{\mathrm{II}^2}\,b^2)} \quad , \quad t_j = \frac{a + i\,\mu_j^{\mathrm{II}}\,b}{a - i\,\mu_j^{\mathrm{II}}\,b} \ .$$

Again the introduced complex quantities C_{jm} and D_{jm} still have to be determined. The characteristic variable ω_j^{II} is the result of a conformal mapping of the outer circular region onto an outer unit circle area and again it specializes to the Euler function on the circular hole boundary. Considering the terms in brackets in (15) for the inner ellipse along the elliptical boundary they take the following form

$$\left(\lambda_j^{\mathrm{II}m} + t_j^m\, \lambda_j^{\mathrm{II}(-m)}\right)|_{ellipse} = \left(1 + t_j^m\right)\, \cos(m\,\phi) + i\, \left(1 - t_j^m\right)\, \sin(m\,\phi) \quad . \tag{16}$$

For the determination of the complex constants A_j, B_{jm}, C_{jm} and D_{jm} in equations (13) and (15) the given boundary conditions at infinity and along the hole edge and in addition the compatibility conditions along the reinforcement boundary have to be considered. The imposed homogeneous bending at infinity

$$M_x \to M_x^\infty \quad , \quad M_y \to M_y^\infty \quad , \quad M_{xy} \to M_{xy}^\infty \quad \text{when } z_j \to \infty \tag{17}$$

yields three real equations for the complex constants A_j:

$$2\,\mathrm{Re}\left[d_j^{\mathrm{I}}\, A_j\right] = M_x^\infty \quad , \quad 2\,\mathrm{Re}\left[e_j^{\mathrm{I}}\, A_j\right] = M_y^\infty \quad , \quad 2\,\mathrm{Re}\left[f_j^{\mathrm{I}}\, A_j\right] = M_{xy}^\infty \quad . \tag{18}$$

Without loss of generality the imaginary part $\mathrm{Im}\, A_2$ of the complex constant A_2 can be set to zero. Then the complex constants A_j can be determined by the three real equations (18). In order to ascertain the remaining constants B_{jm}, C_{jm} and D_{jm} the respective boundary conditions at the hole edge are considered first. Here Kirchhoff's substitute transversal force $\overline{Q}_n^{\mathrm{II}}$ and the normal bending moment M_n^{II} have to vanish:

$$\overline{Q}_n^{\mathrm{II}}|_{hole} = 0 \quad , \tag{19}$$

$$M_n^{\mathrm{II}}|_{hole} = 0 \quad . \tag{20}$$

The substitute transversal force is given by

$$\overline{Q}_n^{\mathrm{II}} = Q_n^{\mathrm{II}} + \frac{\partial M_{nt}^{\mathrm{II}}}{\partial s} = 0 \tag{21}$$

and its integration over ds yields

$$\int_0^s \overline{Q}_n^{\mathrm{II}}\, ds = P^{\mathrm{II}}(s) - P^{\mathrm{II}}(0) + M_{nt}^{\mathrm{II}}(s) - M_{nt}^{\mathrm{II}}(0) = 0 \tag{22}$$

where P^{II} is the integrated transversal force. When the quantities M_n and M_{nt} are traced back to the moments M_x^{II}, M_y^{II} and M_{xy}^{II} relations (22) and (20) take the following form:

$$\begin{aligned}(M_y^{\mathrm{II}} - M_x^{\mathrm{II}})\, \cos\alpha\ \sin\alpha + M_{xy}^{\mathrm{II}}\, (\cos^2\alpha - \sin^2\alpha) + P^{\mathrm{II}} - P^{\mathrm{II}}(0) - M_{nt}^{\mathrm{II}}(0) \\ = 0 \quad ,\end{aligned} \tag{23}$$

$$\begin{aligned}M_x^{\mathrm{II}}\, \cos^2\alpha + M_y^{\mathrm{II}} \sin^2\alpha + 2\, M_{xy}^{\mathrm{II}}\, \cos\alpha\, \sin\alpha \\ = 0 \quad .\end{aligned} \tag{24}$$

Herein the variable α denotes the angle between the x-axis and the normal to the hole edge. Equation (23) multiplied by $\cos\alpha$ and equation (24) multiplied by $\sin\alpha$ are added, and then equation (23) multiplied by $\sin\alpha$ is substracted from (24) multiplied by $\cos\alpha$. Introducing the respective complex representations for M_x^{II}, M_y^{II}, M_{xy}^{II}, P^{II}, $P^{\mathrm{II}}(0) = P^{\mathrm{II}}(z_j^{\mathrm{II}} = r)$ and $M_{nt}^{\mathrm{II}}(0) = M_{xy}^{\mathrm{II}}(z_j^{\mathrm{II}} = r)$ and integrating finally leads to the following relations for the complex potentials along the hole edge:

$$\begin{aligned}2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\, \varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}})\right] + 2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\, e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime\prime}(r)\right]\ x - 2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\, \varphi_j^{\mathrm{II}\prime}(r)\right] = 0\ , \\ 2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}})\right] - 2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\, e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime\prime}(r)\right]\ y - 2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime}(r)\right] = 0\ .\end{aligned} \tag{25}$$

With representations (15) and the relation $z_j^{\mathrm{II}} = r\,(\cos\phi + \mu_j^{\mathrm{II}}\sin\phi)$ along the hole edge equations (25) read:

$$
\begin{aligned}
&2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\left(\sum_{m=1}^{\infty}\{C_{jm}\;(\cos(m\,\phi) - i\;\sin(m\,\phi))\,\} + D_{j0} + \right.\right.\\
&\left.\left. D_{j1}\,r\,(\cos\phi + \mu_j^{\mathrm{II}}\;\sin\phi) + \sum_{n=2}^{\infty}\left\{D_{jn}\left(\lambda_j^{\mathrm{II}^n} + t_j^n\,\lambda_j^{\mathrm{II}(-n)}\right)\right\}\right)\right]\\
&+2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\,e_j^{\mathrm{II}}\,\varphi_j^{\mathrm{II}\prime\prime}(r)\right]\;r\;\cos\phi - 2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\,\varphi_j^{\mathrm{II}\prime}(r)\right] \qquad = 0 \quad ,\\
&2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\left(\sum_{m=1}^{\infty}\{C_{jm}\;(\cos(m\,\phi) - i\;\sin(m\,\phi))\,\} + D_{j0} + \right.\right.\\
&\left.\left. D_{j1}\,r\,(\cos\phi + \mu_j^{\mathrm{II}}\;\sin\phi) + \sum_{n=2}^{\infty}\left\{D_{jn}\left(\lambda_j^{\mathrm{II}^n} + t_j^n\,\lambda_j^{\mathrm{II}(-n)}\right)\right\}\right)\right]\\
&-2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\,e_j^{\mathrm{II}}\,\varphi_j^{\mathrm{II}\prime\prime}(r)\right]\;r\;\sin\phi - 2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\,\varphi_j^{\mathrm{II}\prime}(r)\right] \qquad = 0 \quad .
\end{aligned}
\tag{26}
$$

Along the reinforcement boundary continuity of the deformed plate has to be enforced and thus the slopes $w_{,x}$ and $w_{,y}$ have to be continuous:

$$
\begin{aligned}
w_{,x}^{\mathrm{I}}|_{ellipse} &= w_{,x}^{\mathrm{II}}|_{ellipse} \quad ,\\
w_{,y}^{\mathrm{I}}|_{ellipse} &= w_{,y}^{\mathrm{II}}|_{ellipse} \quad .
\end{aligned}
\tag{27}
$$

In terms of complex potentials this means

$$
\begin{aligned}
2\,\mathrm{Re}\left[\varphi_j^{\mathrm{I}\prime}(z_j^{\mathrm{I}})\right] &= 2\,\mathrm{Re}\left[\varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}})\right] \quad ,\\
2\,\mathrm{Re}\left[\mu_j^{\mathrm{I}}\,\varphi_j^{\mathrm{I}\prime}(z_j^{\mathrm{I}})\right] &= 2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\,\varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}})\right] \quad .
\end{aligned}
\tag{28}
$$

Substitution of the complex potentials (13), (15) taking into account equations (14) and (16) and the representations $z_j^{\mathrm{I}} = a\,\cos\phi + \mu_j^{\mathrm{I}}\,b\,\sin\phi$, $z_j^{\mathrm{II}} = a\,\cos\phi + \mu_j^{\mathrm{II}}\,b\,\sin\phi$ along the reinforcement boundary yields

$$
\begin{aligned}
&2\,\mathrm{Re}\left[A_j\,(a\;\cos\phi + \mu_j^{\mathrm{I}}\,b\;\sin\phi) + \sum_{m=1}^{\infty}B_{jm}\,(\cos(m\phi) - i\;\sin(m\phi))\right] =\\
&2\,\mathrm{Re}\left[\sum_{n=1}^{\infty}\left(C_{jn}\,\omega_j^{\mathrm{II}(-n)}\right) + D_{j0} + D_{j1}\,(a\;\cos\phi + \mu_j^{\mathrm{II}}\,b\;\sin\phi)\right.\\
&\left.+\sum_{m=2}^{\infty}\left(D_{jm}\,\left(1 + t_j^m\right)\;\cos(m\phi) + D_{jm}\,i\;\left(1 - t_j^m\right)\;\sin(m\phi)\right)\right] \quad ,\\
&2\,\mathrm{Re}\left[\mu_j^{\mathrm{I}}\,A_j\,(a\;\cos\phi + \mu_j^{\mathrm{I}}\,b\;\sin\phi) + \mu_j^{\mathrm{I}}\sum_{m=1}^{\infty}B_{jm}\,(\cos(m\phi) - i\;\sin(m\phi))\right] =\\
&2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\left\{\sum_{n=1}^{\infty}\left(C_{jn}\,\omega_j^{\mathrm{II}(-n)}\right) + D_{j0} + D_{j1}\,(a\;\cos\phi + \mu_j^{\mathrm{II}}\,b\;\sin\phi)\right.\right.\\
&\left.\left.+\sum_{m=2}^{\infty}\left(D_{jm}\,\left(1 + t_j^m\right)\;\cos(m\phi) + D_{jm}\,i\;\left(1 - t_j^m\right)\;\sin(m\phi)\right)\right\}\right] \quad .
\end{aligned}
\tag{29}
$$

Along the reinforcement boundary also Kirchhoff's substitute transversal force $\overline{Q}_n$ and the normal moment M_n have to be continuous which means:

$$
\overline{Q}_n^{\mathrm{I}}|_{ellipse} = \overline{Q}_n^{\mathrm{II}}|_{ellipse} \quad , \tag{30}
$$

$$
M_n^{\mathrm{I}}|_{ellipse} = M_n^{\mathrm{II}}|_{ellipse} \quad . \tag{31}
$$

Integration of the substitute transversal forces

$$\overline{Q}_n^{\mathrm{I}} = Q_n^{\mathrm{I}} + \frac{\partial M_{nt}^{\mathrm{I}}}{\partial s} \quad , \quad \overline{Q}_n^{\mathrm{II}} = Q_n^{\mathrm{II}} + \frac{\partial M_{nt}^{\mathrm{II}}}{\partial s} \tag{32}$$

over ds leads to

$$P^{\mathrm{I}}(s) - P^{\mathrm{I}}(0) + M_{nt}^{\mathrm{I}}(s) - M_{nt}^{\mathrm{I}}(0) = P^{\mathrm{II}}(s) - P^{\mathrm{II}}(0) + M_{nt}^{\mathrm{II}}(s) - M_{nt}^{\mathrm{II}}(0) \tag{33}$$

where again P^{I} and P^{II} denote the integrated transversal force. By means of an appropriate tensor transformation for M_{nt} and M_n in terms of the moments M_x^{I}, M_y^{I}, M_{xy}^{I}, M_x^{II}, M_y^{II} and M_{xy}^{II} in cartesian coordinates equations (33) and (31) take the following form:

$$\begin{aligned} &(M_y^{\mathrm{I}} - M_x^{\mathrm{I}}) \cos\alpha \ \sin\alpha + M_{xy}^{\mathrm{I}} (\cos^2\alpha - \sin^2\alpha) + P^{\mathrm{I}} - P^{\mathrm{I}}(0) - M_{nt}^{\mathrm{I}}(0) = \\ &(M_y^{\mathrm{II}} - M_x^{\mathrm{II}}) \cos\alpha \ \sin\alpha + M_{xy}^{\mathrm{II}} (\cos^2\alpha - \sin^2\alpha) + P^{\mathrm{II}} - P^{\mathrm{II}}(0) - M_{nt}^{\mathrm{II}}(0) \end{aligned} \quad , \tag{34}$$

$$\begin{aligned} &M_x^{\mathrm{I}} \cos^2\alpha + M_y^{\mathrm{I}} \sin^2\alpha + 2\, M_{xy}^{\mathrm{I}} \ \cos\alpha \ \sin\alpha = \\ &M_x^{\mathrm{II}} \cos^2\alpha + M_y^{\mathrm{II}} \sin^2\alpha + 2\, M_{xy}^{\mathrm{II}} \ \cos\alpha \ \sin\alpha \end{aligned} \quad . \tag{35}$$

Next relations (34) and (35) are represented in terms of complex potentials in the analogous manner as within the hole conditions (25). Then on the one hand equation (34) multiplied by $\cos\alpha$ and equation (35) multiplied by $\sin\alpha$ are added, and on the other hand equation (34) multiplied by $\sin\alpha$ is substracted from (35) multiplied by $\cos\alpha$. Substitution of the moments M_x, M_y, M_{xy}, $M_{nt}(0) = M_{xy}(z_j = r)$ and the integrated transversal force P by the respective complex relations and integration finally yields:

$$\begin{aligned} &2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{I}}}{\mu_j^{\mathrm{I}}}\ \varphi_j^{\mathrm{I}\prime}(z_j^{\mathrm{I}})\right] + 2\,\mathrm{Re}\left[\mu_j^{\mathrm{I}}\, e_j^{\mathrm{I}}\, \varphi_j^{\mathrm{I}\prime\prime}(a)\right]\ x - 2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{I}}}{\mu_j^{\mathrm{I}}}\ \varphi_j^{\mathrm{I}\prime}(a)\right] = \\ &2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\ \varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}})\right] + 2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\, e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime\prime}(a)\right]\ x - 2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\ \varphi_j^{\mathrm{II}\prime}(a)\right] \quad , \\ &2\,\mathrm{Re}\left[e_j^{\mathrm{I}}\ \varphi_j^{\mathrm{I}\prime}(z_j^{\mathrm{I}})\right] - 2\,\mathrm{Re}\left[\mu_j^{\mathrm{I}}\, e_j^{\mathrm{I}}\, \varphi_j^{\mathrm{I}\prime\prime}(a)\right]\ y - 2\,\mathrm{Re}\left[e_j^{\mathrm{I}}\ \varphi_j^{\mathrm{I}\prime}(a)\right] = \\ &2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\ \varphi_j^{\mathrm{II}\prime}(z_j^{\mathrm{II}})\right] - 2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\, e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime\prime}(a)\right]\ y - 2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\ \varphi_j^{\mathrm{II}\prime}(a)\right] \quad . \end{aligned} \tag{36}$$

Together with representations (13), (14), (15) and (16) and the relations $z_j^{\mathrm{I}} = a\ \cos\phi + \mu_j^{\mathrm{I}}\, b\ \sin\phi$, $z_j^{\mathrm{II}} = a\ \cos\phi + \mu_j^{\mathrm{II}}\, b\ \sin\phi$ along the reinforcement boundary this gives

$$\begin{aligned} &2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{I}}}{\mu_j^{\mathrm{I}}}\, A_j\,(a\ \cos\phi + \mu_j^{\mathrm{I}}\, b\ \sin\phi) + \frac{d_j^{\mathrm{I}}}{\mu_j^{\mathrm{I}}}\sum_{m=1}^{\infty} B_{jm}\,(\cos(m\phi) - i\ \sin(m\phi))\right] \\ &+2\,\mathrm{Re}\left[\mu_j^{\mathrm{I}}\, e_j^{\mathrm{I}}\, \varphi_j^{\mathrm{I}\prime\prime}(a)\right]\ a\ \cos\phi - 2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{I}}}{\mu_j^{\mathrm{I}}}\, \varphi_j^{\mathrm{I}\prime}(a)\right] = \\ &2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\left\{\sum_{n=1}^{\infty}\left(C_{jn}\,\omega_j^{\mathrm{II}(-n)}\right) + D_{j0} + D_{j1}\,(a\ \cos\phi + \mu_j^{\mathrm{II}}\, b\ \sin\phi)\right.\right. \\ &\left.\left.+\sum_{m=2}^{\infty}\left(D_{jm}\ \left(1 + t_j^m\right)\ \cos(m\phi) + D_{jm}\, i\ \left(1 - t_j^m\right)\ \sin(m\phi)\right)\right\}\right] \\ &+2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\, e_j^{\mathrm{II}}\, \varphi_j^{\mathrm{II}\prime\prime}(a)\right]\ a\ \cos\phi - 2\,\mathrm{Re}\left[\frac{d_j^{\mathrm{II}}}{\mu_j^{\mathrm{II}}}\, \varphi_j^{\mathrm{II}\prime}(a)\right] \quad , \end{aligned} \tag{37}$$

$$
\begin{aligned}
&2\,\mathrm{Re}\left[e_j^{\mathrm{I}}\,A_j\,(a\,\cos\phi+\mu_j^{\mathrm{I}}\,b\,\sin\phi)+e_j^{\mathrm{I}}\sum_{m=1}^{\infty}B_{jm}\,(\cos(m\phi)-i\,\sin(m\phi))\right]\\
&+2\,\mathrm{Re}\left[\mu_j^{\mathrm{I}}\,e_j^{\mathrm{I}}\,\varphi_j^{\mathrm{I}\prime\prime}(a)\right]\,b\,\sin\phi-2\,\mathrm{Re}\left[e_j^{\mathrm{I}}\,\varphi_j^{\mathrm{I}\prime}(a)\right]=\\
&2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\left\{\sum_{n=1}^{\infty}\left(C_{jn}\,\omega_j^{\mathrm{II}(-n)}\right)+D_{j0}+D_{j1}\,(a\,\cos\phi+\mu_j^{\mathrm{II}}\,b\,\sin\phi)\right.\right.\\
&\left.\left.+\sum_{m=2}^{\infty}\left(D_{jm}\,\left(1+t_j^m\right)\,\cos(m\phi)+D_{jm}\,i\,\left(1-t_j^m\right)\,\sin(m\phi)\right)\right\}\right]\\
&+2\,\mathrm{Re}\left[\mu_j^{\mathrm{II}}\,e_j^{\mathrm{II}}\,\varphi_j^{\mathrm{II}\prime\prime}(a)\right]\,b\,\sin\phi-2\,\mathrm{Re}\left[e_j^{\mathrm{II}}\,\varphi_j^{\mathrm{II}\prime}(a)\right]\quad.
\end{aligned}
$$

It is now possible to determine the still unknown complex constants B_{jm}, C_{jm} and D_{jm} from the given six equations for a stress free hole edge (26), for the continuous slopes (29) and for the continuous cross-sectional forces (37) along the reinforcement boundary. So far, the set of given equations contains different functional behaviours in angular direction. For a further solution it is appropriate to expand the complex function $\omega_j^{\mathrm{II}(-n)}$ along the reinforcement boundary in relations (29) and (37) and to expand as well the complex function $\lambda_j^{\mathrm{II}^n}+t_j^n\;\lambda_j^{\mathrm{II}(-n)}$ along the hole edge in (26) as Fourier series:

$$
\begin{aligned}
\omega_j^{\mathrm{II}(-n)} &= F_{jn}+\sum_{m=1}^{\infty}\left(F_{jmn}\;\cos(m\,\phi)+G_{jmn}\;\sin(m\,\phi)\right)\quad,\\
\lambda_j^{\mathrm{II}^n}+t_j^n\;\lambda_j^{\mathrm{II}(-n)} &= H_{jn}+\sum_{m=1}^{\infty}\left(H_{jmn}\;\cos(m\,\phi)+K_{jmn}\;\sin(m\,\phi)\right)\quad,
\end{aligned}
\tag{38}
$$

where F_{jn}, F_{jmn}, G_{jmn}, H_{jn}, H_{jmn} and K_{jmn} are the Fourier coefficients which can be calculated in the following way [8]:

$$
\begin{aligned}
F_{jn} &= \frac{1}{2\,\pi}\int_0^{2\,\pi}\omega_j^{\mathrm{II}(-n)}\;d\phi\;,\quad F_{jmn}=\frac{1}{\pi}\int_0^{2\,\pi}\omega_j^{\mathrm{II}(-n)}\;\cos(m\,\phi)\;d\phi\;,\\
G_{jmn} &= \frac{1}{\pi}\int_0^{2\,\pi}\omega_j^{\mathrm{II}(-n)}\;\sin(m\,\phi)\;d\phi\;,\; H_{jn}=\frac{1}{2\,\pi}\int_0^{2\,\pi}\left(\lambda_j^{\mathrm{II}^n}+t_j^n\;\lambda_j^{\mathrm{II}(-n)}\right)\;d\phi\;,\\
H_{jmn} &= \frac{1}{\pi}\int_0^{2\,\pi}\left(\lambda_j^{\mathrm{II}^n}+t_j^n\;\lambda_j^{\mathrm{II}(-n)}\right)\;\cos(m\,\phi)\;d\phi\;,\\
K_{jmn} &= \frac{1}{\pi}\int_0^{2\,\pi}\left(\lambda_j^{\mathrm{II}^n}+t_j^n\;\lambda_j^{\mathrm{II}(-n)}\right)\;\sin(m\,\phi)\;d\phi\;.
\end{aligned}
\tag{39}
$$

Eventually the demand of equality of the resultant coefficients of the trigonometrical functions cosine and sine leads to a series of linear equations for $m=0$, $m=1$ and $m\geq 2$. The dimension and accuracy of the system of equations depends on the chosen range of the indices m and n in the series representations. In the present case $m,\,n=1,\ldots,30$ turns out to be sufficient. Then the complex quantities B_{jm}, C_{jm} and D_{jm} can be determined in a straightforward way and with them the complex representation of all essential field quantities is available everywhere around the reinforced hole.

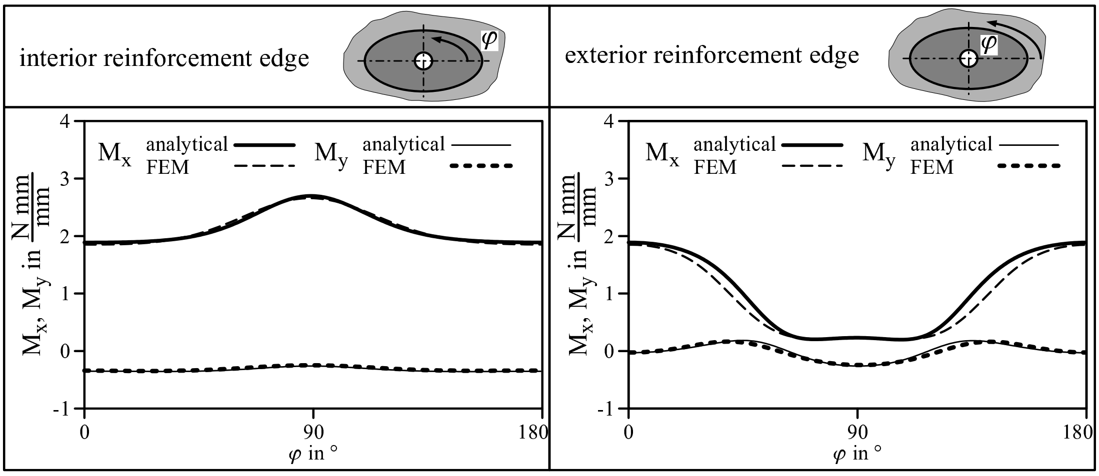

Figure 3: Bending moments M_x and M_y along the interior and exterior reinforcement boundary

NUMERICAL SIMULATION BY THE FINITE-ELEMENT-METHOD

In order to verify the derived analytical solution numerical simulations have been performed by the well-known finite element method, too. The numerical simulations have been performed by the commercial code ANSYS. For the generation of the two-dimensional finite element model 8-node displacement based plate elements with six degrees of freedom have been used. In order to define the laminate layup the total number of plies and the individual ply properties like ply thicknesses, ply angles and engineering constants have to be specified. The hole and reinforcement boundary have been discretized by 300 elements for a sufficient accuracy of the numerical results which has been determined by a respective convergence investigation. The correspondingly fine discretization leads to a significant computational effort.

EXAMPLE AND DISCUSSION

As an actual example a symmetric $[0°/90°]_s$ cross-ply laminate plate with an elliptical reinforcement consisting of a symmetric $[\pm 30°]_s$ angle ply has been considered. Within the reinforcement region this leads to a resultant $[\pm 30°/\mp 30°/0°/90°]_s$ layup. For the material properties of the investigated laminate a T300/Epoxy standard material and an individual ply thickness of $t_p = 0.25\,\mathrm{mm}$ has been defined. This leads to the following non-zero bending stiffnesses:

$$\begin{aligned}
&D_{11}^{\mathrm{I}} = 10001.9\,\mathrm{Nmm}\ , \quad D_{22}^{\mathrm{I}} = 2147.0\,\mathrm{Nmm}\ , \quad D_{66}^{\mathrm{I}} = 416.7\,\mathrm{Nmm}\ ,\\
&D_{12}^{\mathrm{II}} = 226.2\,\mathrm{Nmm}\ ,\\
&D_{11}^{\mathrm{II}} = 187118.6\,\mathrm{Nmm}\ , \quad D_{22}^{\mathrm{II}} = 43111.8\,\mathrm{Nmm}\ , \quad D_{66}^{\mathrm{II}} = 60145.4\,\mathrm{Nmm}\ ,\\
&D_{12}^{\mathrm{II}} = 55003.4\,\mathrm{Nmm}\ , \quad D_{16}^{\mathrm{II}} = 5029.9\,\mathrm{Nmm}\ , \quad D_{26}^{\mathrm{II}} = 1772.6\,\mathrm{Nmm}\ .
\end{aligned} \tag{40}$$

For the geometrical dimensions a hole radius of $r = 20\,\mathrm{mm}$ and aspect ratios of $a = 60\,\mathrm{mm}$ and $b = 50\,\mathrm{mm}$ for the reinforcement patches are assumed. For the applied bending load a remote bending moment $M_{xx}^{\infty} = 1\,\mathrm{N}$ is presumed. As an example Fig. 3 shows the resultant bending moments M_x and M_y along the interior and exterior reinforcement boundary. In addition Fig. 4 shows the bending moments M_x and M_y along the hole edge accordingly. The analytical results are both in quality and numerical quantity in

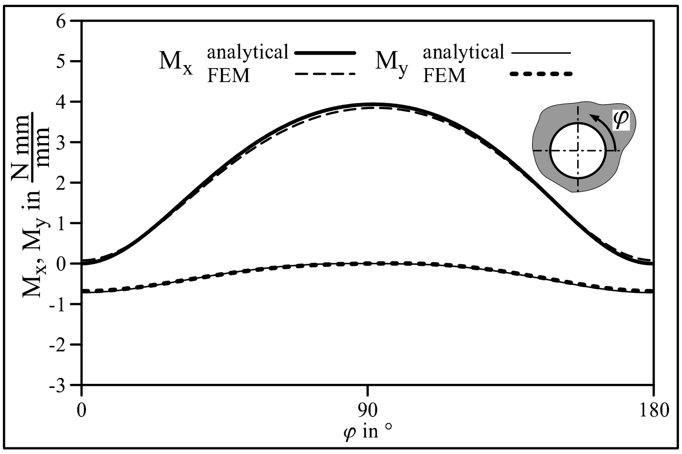

Figure 4: Bending moments M_x and M_y along the hole edge

good correspondence to the numerically determined moment distributions. The necessary CPU-time for the finite element analysis however is about 50 times the CPU-time of the analytical solution and thus the closed-form analytical solution is very effective and useful in regard of computational effort.

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