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THE LINEARIZATION AND ENERGY APPROACH TO THE CONTACT PROBLEM OF COMPOSITE BEAM

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ABSTRACT: The nonlinear contact problem of composite beam is linearized by the inverse method, i.e. the contact pressure distribution and the dimension of contact region are assumed to be given, and the cylinder curvature is to be solved. By means of the principle of superposition, the loading state is decomposed into symmetric and antisymmetric loading states. The mechanics models of these two loading states are further simplified respectively according to the Saint-Venant's principle. The trigonometric series and Legendre series are applied to describing the displacement fields of above two models. The principle of multi-zone generalized potential energy is used to determine the unknown generalized displacements. A family of straight lines (passing through origin) of the indentor curvature versus the external load for different contact region is figured out. Based on these straight lines, the contact length can be determined from the known indentor curvature and the external load. In addition, the glue layers are considered completely the same as the composite plies in analysis. From the computational results, it can be shown that the displacements and stresses converge very well. Furthermore, the shear stress and normal stress obtained from constitutive equations and that obtained from equilibrium equations agree with each other very well.

KEYWORDS: contact problem, composite beam, inverse method, principle of superposition, trigonometric series, Legendre series

1 INTRODUCTION

In the damage tolerance design of composite aircraft structure, it is necessary to consider the contact damage of cover plate caused by the low energy impact from foreign object. Evidently, a correct and simple analysis of contact stress is the prerequisite of contact damage research. It is known that the contact analysis is a difficult task in solid mechanics due to its nonlinearity, especially in the analysis of composite structures. The classical contact theory—Hertz formula [1] is based upon the assumption of semi-infinite plane or space. However, compared to the dimension of contact region, the thickness of each ply in general composite structure is finite. Various methods have been used to research the contact behavior of composite structures [2-5]. But these methods are complex for solving the nonlinear problem straightly, moreover, the thickness and rigidity of glue layer are assumed to be zero and infinite respectively in their analysis. In the future damage mechanics analysis of composite structures, however, it is very important to know the damage coupled stress distributions of glue layer, because many failures of the composite structures due to the delamination caused by shear stress or normal stress in glue layers were encountered. To simplify the analysis, the generalized parabolic contact pressure distribution is applied to simulating the actual one. To avoid the complex nonlinear problem, the inverse method is applied. Namely, the contact region and the contact pressure distribution are assumed to be given and the indentor curvature is considered as unknown. To consider the analysis of delamination damage and fatigue life prediction further, the thickness and rigidity of glue layer are considered, and the energy method is applied instead of exact solution [6] in this paper. The result shows that the displacements and stresses converge rapidly, the shear stress and transverse normal stress obtained from constitution equations agree with that obtained from equilibrium equations very well respectively.

2 DECOMPOSITEION FOR THE LOADING STATE

Fig.1 shows a simply supported composite beam pressed by a cylinder at the mid-point of upper surface. L and $2h$ denote the beam span and height respectively. The curvature of cylinder is κ. The total pressure on the beam is P. The length of contact region is $2a$.

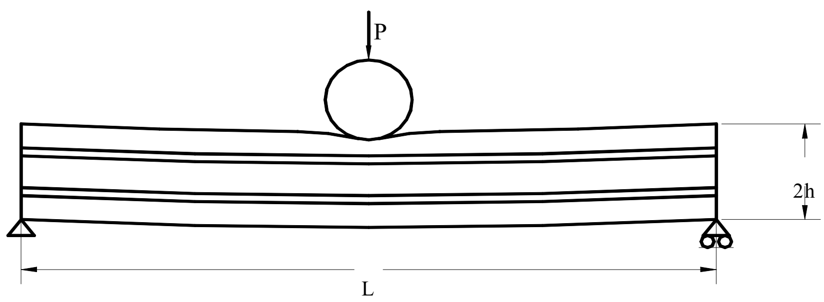


Fig.1 The original loading state of composite beam

The loading state is decomposed into symmetric state $<\alpha>$ and antisymmetric state $<\beta>$, as shown in Fig.2(a) and (b) respectively. These two states can be analyzed separately in simpler ways.

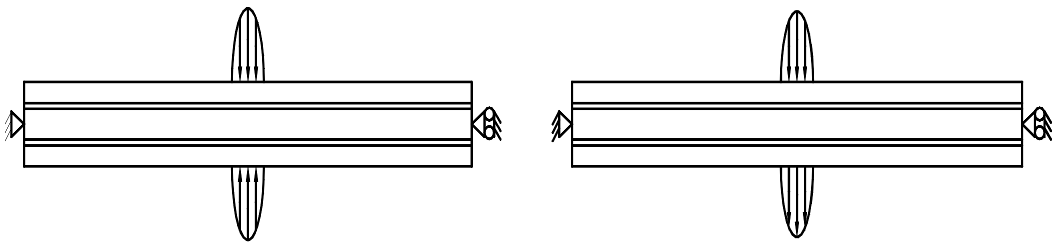


Fig.2 (a) the symmetric loading state $<\alpha>$ Fig.2 (b) the antisymmetric loading state $<\beta>$

3 THE ANALYSIS OF SYMMETRIC LOADING STATE

3.1 The further simplification for the mechanics model of symmetric loading state

Compared to that of composite beam, the length of contact region is very narrow, so the stress field of beam has obvious local effects. According to the Saint-Venant's principle, the contact analysis is only needed in the internal part of beam and the two external ones can be considered as stress free. The load of the internal part with length l, which is determined by the Saint-Venant's principle, is symmetrical about abscissa and ordinate, as shown in Fig.3.

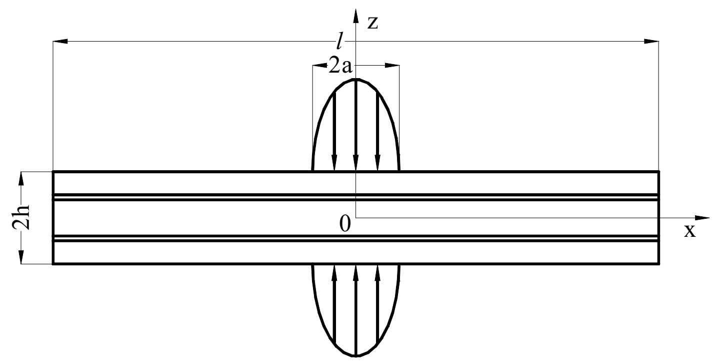


Fig.3 The further simplification for the mechanics model of symmetric loading state

3.2 Displacement field

The beam is subjected to transverse load that is symmetrical about the abscissa and the ordinate. The transverse displacement *w* and longitudinal displacement *u* are symmetrical about the ordinate and the abscissa. So, *w* is the even function of *x* and the odd function of *z*, *u* is the odd function of *x* and the even function of *z*. The displacement field is described as follows

$$\left.\begin{aligned} & w^{(k)}(x,z)=\sum_{m,n} C_{mn}^{(k)} f_m(x) g_n(z) \\ & u^{(k)}(x,z)=\sum_{m,n} A_{mn}^{(k)} s_m(x) t_n(z) \\ & m,n=1,2,3\Lambda \end{aligned}\right\} \tag{1}$$

Where *m* and *n* are the sequence number of each function term in the above series, *k* is the sequence number of every composite ply or glue layer. They have the same meanings in the other equations. A_{mn}^k and C_{mn}^k are the unknown generalized displacements. The function terms of series in the Eq.(1) are

$$f_m(x)=(-1)^{m+1}\cos(\frac{2m-1}{l}\pi x) \qquad s_m(x)=(-1)^m\sin(\frac{2m-1}{l}\pi x)$$

$$g_n(z)=P_{2n-1}(\frac{z}{h})=\sum_i^{n-1}\frac{(-1)^i(4n-2i-2)!}{2^{2n-1}i!(2n-i-1)!(2n-2i-1)!}(\frac{z}{h})^{2n-2i-1}$$

$$t_n(z)=P_{2n-2}(\frac{z}{h})=\sum_i^{n-1}\frac{(-1)^i(4n-2i-4)!}{2^{2n-2}i!(2n-i-2)!(2n-2i-2)!}(\frac{z}{h})^{2n-2i-2}$$

And $P(\frac{z}{h})$ is the Legendre function.

3.3 The contact pressure distribution

The contact pressure distribution is expressed by following function,

$$p(x)=\begin{cases} -p_0[1-(\frac{x}{a})^{\overline{m}}] & x\in[-a,a] \\ 0 & x\notin[-a,a] \end{cases} \quad (2)$$

Where, p_0 is the maximum pressure intensity, a is the half-length of contact region, $\overline{m}$ is determined by the compatibility condition of displacement along the contact surface (the equation is shown in sec. 6). From Eq.(2), the relationship between the total external load P and the maximum pressure intensity p_0 is expressed by the following function

$$P=2\frac{a[1-(-1)^{\overline{m}}+2\overline{m}]}{1+\overline{m}}p_0 \quad (3)$$

The contact load is expanded by the following trigonometric series:

$$p(x)=\sum_{\tilde{m}}p_{\tilde{m}}(-1)^{\tilde{m}+1}\cos(\frac{2\tilde{m}-1}{l}\pi x) \quad (4)$$

Where, $p_{\tilde{m}}=\frac{2}{l}\int_{-a}^{a}p(x)(-1)^{\tilde{m}+1}\cos(\frac{2\tilde{m}-1}{l}\pi x)dx$, and $\tilde{m}$ is the sequence number of trigonometric series.

3.4 Energy-method

Now, the generalized displacements $A_{mn}^{(k)}$ and $C_{mn}^{(k)}$ in Eq.(1) will be determined by the principle of multi-zone generalized potential energy. The strain energy of beam is given by

$$U=\sum_k U_k=\sum_k\frac{1}{2}\int_{z_{k-1}}^{z_k}\int_{-l/2}^{l/2}\sigma_{ij}^{(k)}\varepsilon_{ij}^{(k)}dxdz \quad (5)$$

The constitution relation of *k-th* composite ply or glue layer is

$$\begin{Bmatrix}\sigma_x^{(k)}\\ \sigma_z^{(k)}\\ \tau_{xz}^{(k)}\end{Bmatrix}=\begin{bmatrix}Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)}\\ Q_{21}^{(k)} & Q_{22}^{(k)} & Q_{26}^{(k)}\\ Q_{61}^{(k)} & Q_{62}^{(k)} & Q_{66}^{(k)}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^{(k)}\\ \varepsilon_z^{(k)}\\ \gamma_{xz}^{(k)}\end{Bmatrix} \quad (6)$$

The geometry equation of *k-th* ply or layer is

$$\varepsilon_{ij}^{(k)}=\frac{1}{2}(\frac{\partial u_i^{(k)}}{\partial x_j}+\frac{\partial u_j^{(k)}}{\partial x_i}) \quad (7)$$

From Eq.(1) and (5)~(7), the strain energy of *k-th* ply or layer in composite laminate is expressed as

$$U^{(k)}=\frac{1}{2}\{\sum_{m,n,p,q}S_{kmnpq}^{(AA)}A_{mn}^{(k)}A_{pq}^{(k)}+\sum_{m,n,p,q}S_{kmnpq}^{(AC)}A_{mn}^{(k)}C_{pq}^{(k)}+\sum_{m,n,p,q}S_{kmnpq}^{(CA)}C_{mn}^{(k)}A_{pq}^{(k)}+\sum_{m,n,p,q}S_{kmnpq}^{(CC)}C_{mn}^{(k)}C_{pq}^{(k)}\} \quad (8)$$

where $S_{kmnpq}^{(AA)}$ 、 $S_{kmnpq}^{(AC)}$ 、 $S_{kmnpq}^{(CA)}$ and $S_{kmnpq}^{(CC)}$ are known coefficients determined from Eq.(1), (5)~(7).

The potential energy of external force is given by

$$V=-2\int_{-a}^{a}p(x)w^{(k_{\max})}(x,h)dx-2\int_{-h}^{h}\sigma_x(\frac{l}{2},z)u(\frac{l}{2},z)dz-2\int_{-h}^{h}\tau_{xz}(\frac{l}{2},z)w(\frac{l}{2},z)dz \quad (9)$$

Where, $k_{\max}$ is the sequence number of the outmost composite ply.

On the interface between the *k-th* layer and the *k+1-th* layer，the conditions of continuity must be satisfied by the displacement and the surface traction. Then, an additional term should be included in the functional of total potential energy as following

$$J^{(k,k+1)} = \int_{S^*} [(u^{(k+1)} - u^{(k)})\tau_{xz}^{(k)} + (w^{(k+1)} - w^{(k)})\sigma_z^{(k)}]dS^* \tag{10}$$

Based on the principle of multi-zone generalized potential energy, we obtain

$$\delta\Pi = \delta\left\{\sum_{k=1}^{k_{\max}}(U^{(k)} + J^{(k,k+1)}) + V\right\} = 0 \tag{11}$$

According to the Eq(11), a system of equations about the generalized displacements $A_{mn}^{(k)}$ and $C_{mn}^{(k)}$ is obtained as follows

$$\left.\begin{array}{l} \sum\limits_{k,p,q} T_{kmnpq}^{(AA)} \cdot A_{pq}^{(k)} + \sum\limits_{k,p,q} T_{kmnpq}^{(AC)} \cdot C_{pq}^{(k)} = 0 \\ \sum\limits_{k,p,q} T_{kmnpq}^{(CA)} \cdot A_{pq}^{(k)} + \sum\limits_{k,p,q} T_{kmnpq}^{(CC)} \cdot C_{pq}^{(k)} = B_{kmn} \end{array}\right\} \tag{12}$$

(k=1,2,…kmax，m=1,2,3…M，n=1,2,3…N)

Where, the value of M is the maximum number of $\tilde{m}$ which is obtained from Eq.(4), the value of N is determined by the convergence condition of stresses. $T_{kmnpq}^{(AA)}$, $T_{kmnpq}^{(AC)}$, $T_{kmnpq}^{(CA)}$, $T_{kmnpq}^{(CC)}$ and B_{kmn} are known coefficients deduced from Eq.(1) ~(11). $A_{mn}^{(k)}$ and $C_{mn}^{(k)}$ are determined by solving the above equations. Then the displacement components, strain and stress components are obtained also.

4 THE ANALYSIS OF ANTISYMMETRIC LOADING STATE

4.1 The further simplification for the mechanics model of antisymmetric loading state

Then we discuss the stress field of composite beam under the antisymmetric load. According to the Saint-Venant's principle, the contact analysis is only needed in the internal part of beam and the external parts can be analyzed by classical beam theory. Furthermore, the pure bending state can be analyzed by classical beam theory also, so it is only needed to study the loading state shown in Fig.4.

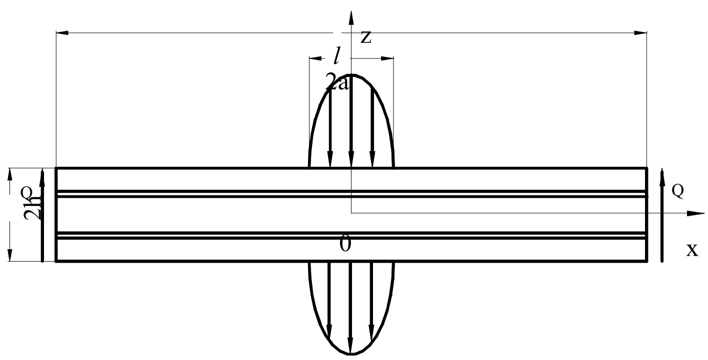


Fig.4 The further simplification for the mechanics model of antisymmetric loading state

4.2 The solutions for the antisymmetric loading state $< \beta >$

The beam is subjected to the transverse load which is symmetric about the ordinate and antisymmetric about the abscissa. So, w is the even function of x and z, and u is the odd function of them. The displacement expressions are the same with Eq.(1), however, $t_n(z)$ and $g_n(z)$ must be substituted by the following expressions.

$$\left.\begin{aligned} g_n(z) = P_{2n-2}(\frac{z}{h}) = \sum_{i}^{n-1} \frac{(-1)^i(4n-2i-4)!}{2^{2n-2} i!(2n-i-2)!(2n-2i-2)!}(\frac{z}{h})^{2n-2i-2} \\ t_n(z) = P_{2n-1}(\frac{z}{h}) = \sum_{i}^{n-1} \frac{(-1)^i(4n-2i-2)!}{2^{2n-1} i!(2n-i-1)!(2n-2i-1)!}(\frac{z}{h})^{2n-2i-1} \end{aligned}\right\} \tag{13}$$

To solve the unknown generalized displacements in the displacement field expressions of this load state, it is just needed to take the same steps that have been done in the section 3.

5 THE DEFORMATION OF CYLINDER

The cylinder can be considered as a semi-plane when the dimension of contact region is much smaller than the radius of cylinder. From the theory of elasticity [7], the transverse displacement on the contact surface of cylinder is

$$w(x,h) = -\frac{2h}{\pi E}\int_{-\frac{a}{h}}^{\frac{a}{h}} 2p(\frac{\xi}{h}) \cdot \ln(\frac{|x-\xi|}{h})d(\frac{\xi}{h}) \tag{14}$$

Where, $|x-\xi|$ is the positive distance between the load components on the point of ξ and x.

6 THE CURVATURE OF CYLINDER

On the basis of the least-square method and the results in section 3~5, the following equation is studied.

$$\Delta^2 = \int_0^a \left\{(w_1(0,h) - w_1(x,h)) + (w_2(0,h) - w_2(x,h)) + \frac{1}{2}\kappa x^2\right\}^2 dx \tag{15}$$

where $w_1(0,h)$ and $w_2(0,h)$ are the deflections of beam and cylinder respectively at the mid-point in the contact region, $w_1(x,h)$和 $w_2(x,h)$ are the deflections of beam and cylinder respectively at the arbitrary point in the contact region, κ is the curvature of cylinder. Then $\overline{m}$ and the corresponding value of κ are determined by the minimum conditions of Eq.(15).

7 THE SOLUTION OF CONTACT PROBLEM BY INVERSE METHOD

Through the process of the section 2~6, the exponent $\overline{m}$ and the corresponding value of curvature κ for the given contact length a and the total external load P can be obtained. The curvature κ is in direct proportion to the total load P for the given contact length, that is to say, we can obtain a κ versus P curve (the straight line passing through origin) when the contact length is given. Then we can figure out a family of straight lines for the different contact length. From these straight lines, we can obtain a family of a versus P curves for the different curvature κ. In practical application, the contact region is determined from the above family of curves. Further, the displacement and stress fields are determined also. Fig.9~10 are the typical examples of above curves used to determine the contact region.

8 EXAMPLE

To verify the availability of above method, a typical example is presented.

Each composite ply is isotropic transversally in the section that is perpendicular to the fiber. The constants are given by $E_1=E_0$, $E_2=0.1E_0$, $\mu_1=0.30$, $\mu_2=0.03$, $G_{12}=0.035\,E_0$. The beam span $L=40h$, the width of beam $B=0.4h$ and the height $H=2h$. The length of Saint-Venant's effective region $l=8h$. The composite laminates are symmetrical about the abscissa, and the lay-up of the plate is [$0^0/90^0/0^0$]. The thickness of each laminate is [$0.4875h/0.975h/0.4875h$]. The thickness of every glue layer is $0.025h$.

The glue layer is isotropic and the material constants are given by $E_g=0.021E_0$, $\mu_g=0.30$.

The indentor is isotropic and the material constants are assumed to be $E_c=E_0$, $\mu_c=0.30$.

The external load is $P=1.5e\text{-}4\,E_0h^2$. The length of contact region is $2a=h$.

The convergence tests of deflection, stress and curvature of cylinder are shown in table 1.

The stress distributions are obtained also. They are shown in Fig. 5~8.

Table 1 The convergence tests of deflection, stress of beam and the curvature of cylinder

N	$w(0,h)/h$	$h\kappa$	$\sigma_x(0,h)/E_0$	$\sigma_z(0,h)/E_0$	$\tau_{xz}(l/4,h/2)/E_0$
1	-.6911938E-01	.3745048E-01	-.1174951E-02	-.3076143E-03	.0000000E+00
2	-.6193465E-01	.3709617E-01	-.2273542E-02	-.5886364E-03	.5238619E-04
3	-.6203699E-01	.3777104E-01	-.2655531E-02	-.6345951E-03	.9930677E-04
4	-.6209834E-01	.3851703E-01	-.2929694E-02	-.6081321E-03	.1139361E-03
5	-.6210224E-01	.3859236E-01	-.2946044E-02	-.5936485E-03	.1132410E-03
6	-.6210574E-01	.3865156E-01	-.2945707E-02	-.5946558E-03	.1131317E-03
7	-.6210530E-01	.3863740E-01	-.2928500E-02	-.5881171E-03	.1131213E-03
8	-.6210518E-01	.3863303E-01	-.2922798E-02	-.5910725E-03	.1131258E-03
9	-.6210509E-01	.3863036E-01	-.2919447E-02	-.5893117E-03	.1131240E-03
10	-.6210506E-01	.3862946E-01	-.2919007E-02	-.5905302E-03	.1131246E-03

Note: the σ_x in above table does not include the contribution of pure bending state.

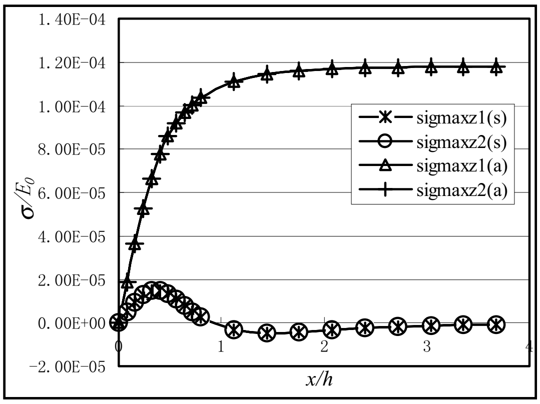


Fig.5 transverse shear stress distribution along x axis on the section of $z=h/2$

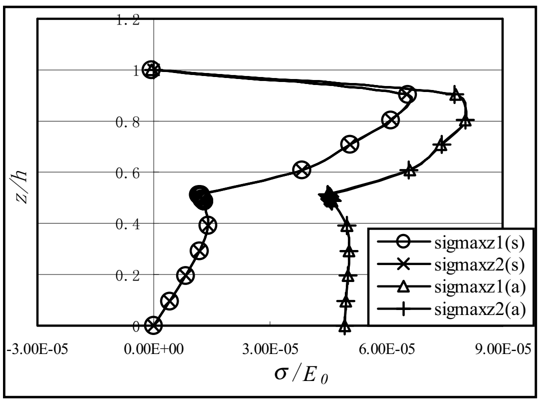
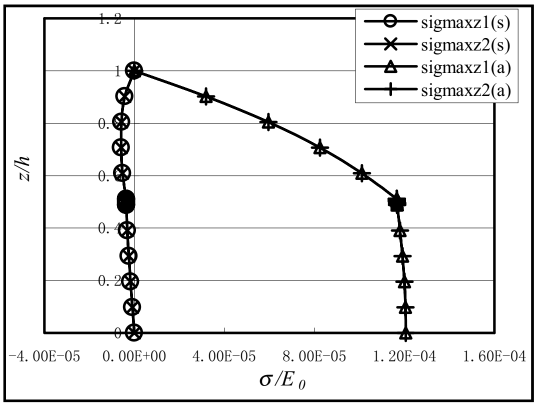


Fig.6 transverse shear stress distribution along z axis on the section of $x=2a/5=l/40$



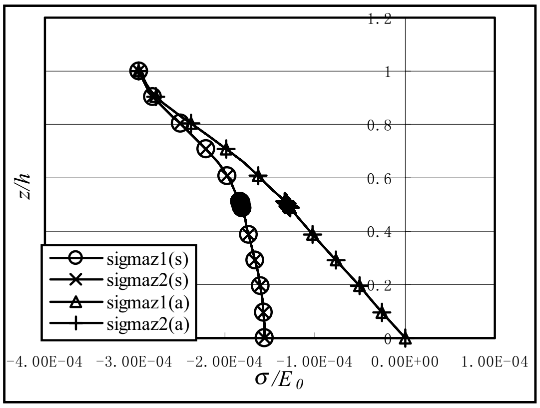
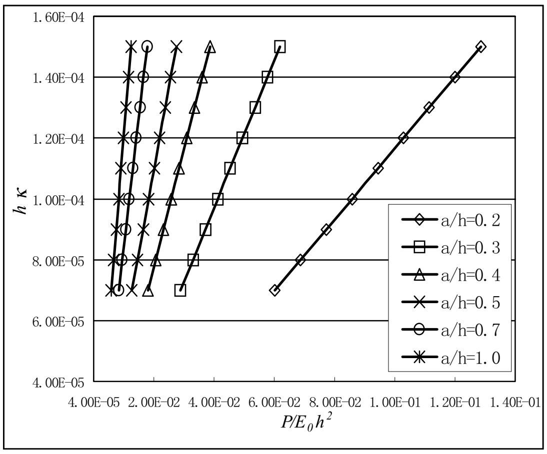


Fig.7 transverse shear stress distribution along z axis on the section of *x=4a=l/4*

Fig.8 transverse normal stress distribution along z axis on the section of *x=0*

Note: About the designations for each curves in the Fig.5~8, '1' denotes the stress obtained from the constitution equation, '2' denotes the stress obtained from the equilibrium equation, 's' denotes the stress of symmetric loading state, 'a' denotes the stress of antisymmetric loading state.



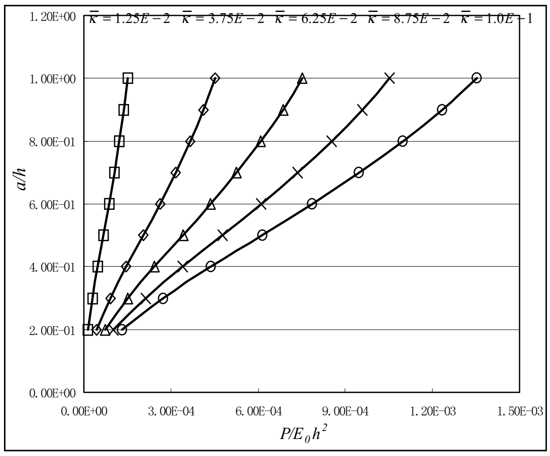


Fig.9 $\overline{\kappa}$ versus $\overline{P}$ curves for different contact length

Fig.10 $\overline{a}$ versus $\overline{P}$ curves for different curvature of cylinder

Note: In above figures, $\overline{a} = a/h$, $\overline{P} = P/E_0h^2$, $\overline{\kappa} = h \cdot \kappa$

9 CONCLUSIONS

From the above analysis, the following conclusions are reached:

(1) The nonlinear contact problem of composite beam can be linearized by inverse method, that is to say, the contact pressure distribution and the contact region are assumed to be given, the curvature of cylinder is to be solved.

(2) The problem is analyzed in simpler ways after being decomposed into symmetric and antisymmetric states by means of the principle of superposition.

(3) According to the Saint-Venant's principle, the contact load is expanded by trigonometric series in few terms.

(4) The trigonometric series and Legendre series are applied to describing the displacement field of above two loading states advantageously, and the principle of

multi-zone generalized potential energy is used to determine the unknown generalized displacements effectively.

(5) The results show that the displacements and stresses converge rapidly, the shear stress and transverse normal stress obtained from constitution equations agree with that obtained from equilibrium equations very well respectively. The above results also validate the approach in this paper.

(6) Through the inverse method, we can obtain many families of the contact region versus the indentor curvature curves for the different external load and the contact region versus the total load curves for the different indentor curvature. Based on these families of curves, the contact region can be determined from the known indentor curvature and the external load.

(7) This methodology can be extended to the analysis of contact behavior of composite plates.

(8) This model can be applied to the damage analysis of delaminated failure.

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