Determination of Centroid and Shear Center Locations of Composite Box Beams

W. S. Chan and K.A. Syed
The University of Texas at Arlington
Arlington, Texas, USA
Email:chan@uta.edu

SUMMARY
This paper presents a simple method based on classical lamination theory to determine the locations of the centroid and the shear center for composite beams with box cross-section. The present method includes the effect of coupling due to unsymmetrical cross-section and its laminate layup while the conventional method using the smear properties of laminate ignores this effect.

Keywords: Composite structures; Hat section beam; sectional stiffness; shear center; centroid

INTRODUCTION
There has been a growing interest in the analysis of thin-walled composite beams because of increasing use of composite structures in aerospace and civil engineering applications. In structural applications, structural member with thin-walled cross-section is one of the most efficient structural members that can achieve the required stiffness with minimum weight. In analyzing these structures, most structures are analyzed as beams as one-dimensional structural members if one dimension (the length) is much larger than the other two dimensions (width and thickness).

The foundation of the beam analysis is based upon the moment-curvature relationship along the longitudinal axis of the beam. This approach used for laminated composite beam is not different from the isotropic beam. However, in evaluation of the relationship between the bending/twisting moment and the curvatures, so-called the bending and torsion stiffnesses of the laminated composite beam possesses a unique behavior that is different from the isotropic beam. Geometric properties such as the locations of the centroid and the shear center of the cross-section are often used in evaluation of the structural response. For beams made of isotropic material, these locations are dependent of geometry of the cross-section only but not the material property.

The analysis of composite beam has been extensively studied for sometimes. Several books that contain composite beam analysis were published [1-6]. Analytical methods presented in those books are not simple enough to be used in design practice. On the other hand, when evaluating sectional property of the beam, smear property of structural laminates are used. In doing so, the coupling effect among axial, bending and twisting due to unsymmetrical layup and unsymmetrical cross-section are not included. In analysis of
composite tubular beams, Chan and his co-workers [7, 8] include ply orientation change along the beam section contour in formulating their stiffness model. Their results indicated that using smear property for computing bending stiffness of composite tubular section can results in significant error in bending stiffness. Recently, Syed and Chan [9] evaluated sectional stiffness of laminated composite beam with hat-section. In their method, the stiffness for a structural cross-section is converted into the stiffness for a thin plate section. Then the lamination theory is applied for calculating the mid-plane strain and curvature of the plate. The ply stress can then be obtained for any specified point in the original structural cross-section. Syed et al. [10] included the thermal effect in the stress analysis of the hat-section.

Although the computer capability has been tremendously increased in the past decades, analysis by using FEM for laminated composite beams is still not an efficient method because of structural configuration dependent. Hence, there is a need for analytical methods that not only provide acceptable accuracy in evaluation of sectional property for better prediction of structural response but also can be easily used for parametric study. The overall purpose of this study aims to development of an approximate analytical method for engineering practice in analyzing the response of composite box beams. The analytical method is simply to be used and easily to be used for parametric study without loss of the great accuracy.

**CENTROID AND SHEAR CENTER**

Centroid and shear center of a beam are two important sectional properties that are used to determine the structural response. The centroid is defined as the location where an axial load does not cause a change in curvature and a bending moment does not produce axial strain. Likewise, shear forces acting at the shear center do not cause twist. In other words, the load acting at the centroid decouples the structural response between axial extension and bending, where as, shear center decouples bending and twisting, mechanisms of a beam. For the beam made of isotropic material, the locations of centroid and shear center are purely dependent of geometric cross-section of the beam. Composite material exhibits unique structural coupling characteristic. For example, a composite beam with rectangular cross-section subjected to axial force exists an extension/twist coupling and applied pure bending moment exists a bending/twisting coupling. Hence, the location of centroid and shear center of a laminated composite closed-section beam is not only a function of the geometry but also the material properties and layup of the sectional laminate. Determination of these locations depends on the stiffness of each segment laminate of the entire cross-section. The following is a brief description of stiffness of the box cross-section.

**CONSTITUTIVE EQUATION OF BOX CROSS-SECTION**

Evaluation of the stiffness of composite beam with hat section has been derived in Ref. 11. The procedure to obtain of the stiffness is briefly described below.

**Geometry of Box-section**
The cross-section of the beam is divided into six laminates according to the shape of the geometry. As shown in Figure 1, the laminates are designated as shown below:

\[ b_{tf} = \text{the width of top flange}, \quad b_{bf} = \text{the width of bottom flanges}, \]
\[ t_{ply} = \text{the thickness of each ply}, \quad b_w = \text{the width of web}, \]
\[ H = \text{the vertical height}. \]

It should be noted that \( b_{tf}, b_{bf} \) and \( H \) are kept constant, where as \( b_w \) varies according to web angle, \( \sin^{-1}(H/b_w) \). Hence, \( b_w \) is dependent of the angle. Two cases of laminate layups shown below are used for this study.

**Case 1: Symmetric Layup**
- Top Flange: \([\pm45/0]_s\)
- Bottom Flange: \([\pm45/0]_s\)
- Web Laminate: \([\pm45]_s\)

**Case 2: unsymmetrical Layup**
- Top Flange: \([\pm45/0]_s\)
- Bottom Flange: \([0_2/45/-45_2/45]_T\)
- Web Laminate: \([\pm45]_s\)

Figure 1 Geometry of Box Cross-section.

### Stiffness Matrices of Box-section

As mentioned, our approach is to convert the stiffness of the cross-section into the stiffness for an equivalent thin plate at the selected reference axis. In Ref. 9, Syed and Chan developed equivalent stiffness matrices for the entire hat cross-section. The matrices contain \([A\], \[B\], and \[D\] matrices that represent extensional, extensional/bending coupling and bending stiffness for this cross-section. A brief description of the procedure is reviewed here. First, the stiffness matrices for each segment are calculated. Then, these stiffness matrices are translated to the common reference axis according to the parallel theorem. For the web segment as shown in Fig. 2, the stiffness matrices are rotated at the inclination angle of the web about the longitudinal axis before its translation. The overall stiffness matrices are obtained by summing the stiffness matrices for each segment taking into account the weighting factor of its width. The results from this stiffness model were in excellent agreement with the FEM results. Evaluation of stiffness matrices of top and bottom flange laminates is straight-forward.

#### Stiffness of web laminate

For the web laminate as shown in Figure 2, the ply stiffness matrix of each play in an infinitesimal element is rotated about \(x\)-axis. The angle ply stiffness in the infinitesimal element can be obtained by rotating \([Q_{1-2}]\) with its fiber orientation, \(\beta\) angle around the 3- (or \(z'\)) axis. The infinitesimal element of the web laminate is then rotated a \(\theta\) (the
web angle) respect to the x’-axis. The overall ply stiffness in the x-y-z coordinates can be written as:

\[
\begin{bmatrix}
\overline{Q}^{''}\end{bmatrix}_{x-y, 0} = \begin{bmatrix}
T_{\sigma}(\theta)\end{bmatrix}_x \cdot \begin{bmatrix}
T_{\sigma}(-\beta)\end{bmatrix}_z \cdot \begin{bmatrix}
Q_{1-2}\end{bmatrix} \cdot \begin{bmatrix}
T_{e}(\theta)\end{bmatrix}_x
\end{bmatrix}_{x}
\]  

(1)

\([T_{\sigma}] \) and \([T_{e}] \) in the above equation are the stress and strain transformation matrices, respectively. The subscripts, x and z in \([T_{\sigma}] \) and \([T_{e}] \) indicate the axes where the stiffness matrix is rotated. The expression of \([T_{\sigma}] \) and \([T_{e}] \) matrices are given in Appendix.

\([A’’]_w, [B’’]_w\) and \([D’’]_w\) are the stiffness matrices per unit width and are given as

\[
\begin{align*}
[A’’]_w &= \sum_{k=1}^{n} \left(\overline{Q}^{''}\right)(z^k_k - z^k_{k-1}) \\
[B’’]_w &= \frac{1}{2} \sum_{k=1}^{n} \left(\overline{Q}^{''}\right) \left(z^2_k - z^2_{k-1}\right) \\
[D’’]_w &= \frac{1}{3} \sum_{k=1}^{n} \left(\overline{Q}^{''}\right) \left(z^3_k - z^3_{k-1}\right)
\end{align*}
\]  

(2)

The subscript, w shown in the above equation refers to the web laminate. The total stiffness matrices of the web laminate can be integrated from \(-b_w/2\) to \(b_w/2\) as shown below. The detailed derivation can be referred to Ref.11.

\[
\begin{align*}
[A]_w &= b_w \cdot [A’’]_w \\
[B]_w &= b_w \cdot [B’’]_w \\
[D]_w &= b_w \cdot [D’’]_w + \frac{b_w^3}{12} \cdot \sin^2 \theta \cdot [A’’]_w
\end{align*}
\]  

(3)

Figure 2 Infinitesimal Section of Web Laminate Stiffness
Constitutive Equation of Box Cross-section

The constitutive equation for a box section can be written as
\[
\begin{bmatrix}
\bar{N} \\
\bar{M}
\end{bmatrix} = \begin{bmatrix}
\bar{A} & \bar{B} \\
\bar{B} & \bar{D}
\end{bmatrix} \begin{bmatrix}
\bar{\varepsilon}_0 \\
\bar{\kappa}
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
\bar{\varepsilon}_0 \\
\bar{\kappa}
\end{bmatrix} = \begin{bmatrix}
\bar{a} & \bar{b}
\end{bmatrix}^T \begin{bmatrix}
\bar{N} \\
\bar{M}
\end{bmatrix}
\] (4)

Where \(\bar{\varepsilon}_0\) and \(\bar{\kappa}\) are the strains and curvatures respect to the reference axis of the entire cross-section. \(\bar{N}\) and \(\bar{M}\) are the applied load and moment matrices of the cross-section.

CENTROID AND SHEAR CENTER OF BOX CROSS-SECTION

Centroid

The centroid is located such that the beam’s axis remains straight when an axial force is applied at the centroid. Although this axis remains straight, the beam may twist about the axis of twist, which does not necessarily coincide with the axis passing through the centroid.

For a narrow beam subjected to an axial load and bending load, \(\bar{\kappa}_y\) and \(\bar{\kappa}_{xy}\) are induced but not moments, \(\bar{M}_y\) and \(\bar{M}_{xy}\). Hence, Eq. 4 can be written as
\[
\begin{bmatrix}
\bar{\varepsilon}_x \\
\bar{\kappa}_x
\end{bmatrix} = \begin{bmatrix}
\bar{a}_{11} & \bar{b}_{11} \\
\bar{b}_{11} & \bar{d}_{11}
\end{bmatrix} \begin{bmatrix}
\bar{N}_x \\
\bar{M}_x
\end{bmatrix}
\] (5)

Let \(z_c\) be the distance measured from the reference axis to the centroid axis. At this location, the strain along the beam axis is zero. Hence, \(z_c\) can be determined as
\[
\bar{\varepsilon}_x = \bar{\varepsilon}_0 + z_c \cdot \bar{\kappa}_x = 0 \quad \text{or} \quad z_c = -\frac{\bar{\varepsilon}_0}{\bar{\kappa}_x}
\] (6)

At the centroid axis, \(\bar{\varepsilon}_x^0\) depends only of \(\bar{N}_x\) and \(\bar{\kappa}_x\) depends only on \(\bar{M}_x\). Therefore, for applied pure moment, \(\bar{N}_x = 0\), Eq. 5 reduces to
\[
\bar{\varepsilon}_x = \bar{b}_{11} \cdot \bar{M}_x \quad \text{and} \quad \bar{\kappa}_x = \bar{d}_{11} \cdot \bar{M}_x
\] (7)

The location of centroid can be obtained from Eqs. 6 and 7 as
\[
z_c = \frac{\bar{b}_{11}}{\bar{d}_{11}}
\] (8)

If the entire cross-section is symmetric with respect to its reference axis including its geometry and its segment laminate layup, then \(\bar{b}_{11}\) is zero. As a result, \(z_c = 0\). This implies that the centroid is located at its reference axis. It should be noted that both \(\bar{b}_{11}\) and \(\bar{d}_{11}\) are dependent on the material constants, ply fiber orientation as well as the stacking sequence.
Shear Center

As stated before, transverse load acting at the shear center, the beam does not cause twist and vice versa. For a narrow beam subjected to a shear, Eq. 4 can be written as,

\[
\begin{bmatrix}
\gamma_{xy} \\
\kappa_{xy}
\end{bmatrix} = \begin{bmatrix}
\bar{a}_{66} & \bar{b}_{66} \\
\bar{b}_{66} & \bar{d}_{66}
\end{bmatrix} \begin{bmatrix}
\bar{N}_{xy} \\
\bar{M}_{xy}
\end{bmatrix}
\]

(9)

Let \( z_{sc} \) be the distance measured from the reference axis to the shear center. At this location, the shear strain along the beam axis is zero. Hence, \( z_c \) can be determined as

\[
\gamma_{xy} = \gamma_{xy}^0 + z_{sc} \cdot \kappa_{xy} = 0 \quad \text{or} \quad z_{sc} = -\frac{\gamma_{xy}}{\kappa_{xy}}
\]

(10)

At shear center, \( \gamma_{xy}^0 \) depends only of \( \bar{N}_{xy} \) and \( \kappa_{xy} \) depends only on \( \bar{M}_{xy} \). Therefore, for \( \bar{N}_{xy} = 0 \), Eq. 7 reduces to,

\[
\gamma_{xy}^0 = \bar{b}_{66} \cdot \bar{M}_{xy} \quad \text{and} \quad \kappa_{xy} = \bar{d}_{66} \cdot \bar{M}_{xy}
\]

(11)

In lieu of Eqs. 10 and 11, we obtain

\[
\beta_{sc} = \frac{\bar{b}_{66}}{\bar{d}_{66}}
\]

(12)

For a symmetric section, \( z_{sc} = 0 \). This implies that the shear center is located at the reference axis. Likewise the previous case, both \( \bar{b}_{66} \) and \( \bar{d}_{66} \) are dependent on the material constants, ply fiber orientation as well as the stacking sequence.

Validation of the present approach

The locations of the centroid and the shear center obtained by the present approach are validated by using conventional approach for the aluminium cross-section. Box sections with five different web angles ranging from 30° to 90° with an interval of 15° were used for study. For all of the sections, the location of the centroid of the box section obtained from both methods is less than 1%. However, comparison of the shear center location is ranging from 0% for the section with 90° web angle to 7.6% for the section with 30° web angle.

Case Study

Two cases are studied for composite box cross-sections which are made of S4/3501-6 graphite/epoxy laminates. The laminate layup of each segment is shown in Fig. 1. In this study, the geometric parameters, \( b_{tf} = 1 \) in, \( H = \) in. shown in Fig. 1 are selected. For a given web angle and the value of \( H \), \( b_{bf} \) and \( b_w \) can be determined.

Case 1: Symmetric Layup

Table 1 lists the calculated results of the centroid and the shear center locations of the beam with five different web angles. The location shown is the distance from the mid-axis of the bottom laminate.
Table 1 Centroid and shear center locations for symmetric lay-up of box beams.

<table>
<thead>
<tr>
<th>Web Angle, °</th>
<th>Centroid Location from Mid-plane of Bottom Laminate (in), ( z_c )</th>
<th>Shear Center Location from Mid-plane of Bottom Laminate (in), ( z_{sc} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>0.5000</td>
<td>0.5000</td>
</tr>
<tr>
<td>75°</td>
<td>0.3977</td>
<td>0.4385</td>
</tr>
<tr>
<td>60°</td>
<td>0.3276</td>
<td>0.3885</td>
</tr>
<tr>
<td>45°</td>
<td>0.2728</td>
<td>0.3458</td>
</tr>
<tr>
<td>30°</td>
<td>0.2232</td>
<td>0.3081</td>
</tr>
</tbody>
</table>

As shown, for a cross-section with 90° web angle, the entire cross-section is fully symmetric. It becomes a square cross-section. The locations for the centroid and shear center coincide exactly at its geometric center. As the web angle decreases, the locations move closer to the bottom flange laminate.

**Case 2: Unsymmetrical Layup**

For this study, only the layup of the bottom flange laminate is unsymmetrical. The rest of laminates are identical to Case 1 (see Fig. 1). The calculated results of the centroid and shear center locations are listed in Table 2.

Table 2 Centroid and shear center locations for unsymmetrical lay-up of box beam

<table>
<thead>
<tr>
<th>Web Angle, °</th>
<th>Centroid Location from Mid-plane of Bottom Laminate (in)</th>
<th>Shear Center Location from Mid-plane of Bottom Laminate (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>0.4970</td>
<td>0.5010</td>
</tr>
<tr>
<td>75°</td>
<td>0.3941</td>
<td>0.4398</td>
</tr>
<tr>
<td>60°</td>
<td>0.3236</td>
<td>0.3900</td>
</tr>
<tr>
<td>45°</td>
<td>0.2686</td>
<td>0.3475</td>
</tr>
<tr>
<td>30°</td>
<td>0.2187</td>
<td>0.3100</td>
</tr>
</tbody>
</table>

As indicated, the location of centroid moves more toward to the bottom flange laminate from the reference axis comparing to the symmetrical case. However, the shear center moves less from its reference axis.

**CONCLUSION**

A simple method based upon lamination theory to determine the locations of centroid and shear center for a laminated box beam is presented. As shown, unlike isotropic material, both centroid and shear center locations dependent of material constants, ply fiber orientation as well as the stacking sequence. The present method was validated by using aluminium beam. Beams with five web angles in both symmetric and unsymmetrical layups were studied. For a symmetrical laminate layup of box beams, the both centroid and shear center locations move toward the bottom flange laminates as
web angle is increased. For box beam with 90° web angle (a squared box beam), locations of centroid and shear center are coincided at its geometric center of cross-section. For an unsymmetrical laminate layup of box beam, the centroid location is closer to the bottom flange laminate comparing to the symmetric case. However, the shear center location is farther from the bottom flange laminate.

References


2. Vasiliev, V. V. and R. M. Jones, 1993, Chapter 4 in Mechanics of Composite Structures, Taylor & Francis, Washington, DC.


