# CALCULATION OF STRESS INTENSITY FACTORS WITH THE MODIFIED VIRTUAL CRACK CLOSURE TECHNIQUE

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## Abstract:

The modified equations of VCCT with different element lengths in front of the crack tip and behind are given out in the paper. In order to avoid the complex post proceeding to extract fracture parameters such as SERR and SIFs, the interface crack element is developed. The interface crack element can be implemented easily in the commercial software ABAQUS™ by the user subroutine UEL. Several examples are analyzed to demonstrate the accuracy of the present method with agreement with analytical solutions.

## 1 Introduction

Delamination is one of the most common failure mode of composite structure [1-2], and fracture mechanics is one effective tool to characterize the onset and growth of delamination [3]. As the basic fracture parameters, SERR ( $G$ ) and SIF ( $K$ ) need to be calculated. Up to now, the two kinds of numerical approaches are widely used: one is called direct method, such as stress extrapolation method and displacement extrapolation method; the other is indirect method, including J integral、interaction integral、VCCT and et al [4-5].

In contrast with other methods, VCCT has many merits such as simplicity、convenience、high accuracy、no sensitivity to mesh size、explicit separation of fracture modes and et al, therefore, it is widely investigated by scientists and engineers [2,6]. In previous research, the equations have been derived under the assumption that the element lengths in front of the crack tip and behind are identical. However, once automatic mesh generators are used to create complex models, especially in the situation of grid transition, the ideal case of identical element length can no longer be assumed and corrections are required.

The study about 2D-VCCT with different element lengths in front of the crack tip and behind is very limited. Based on the assumption that the stress distribution is same at the crack tip before the crack growth and after, a kind of modified equations is derived by Rybichi and Kanninen [7]. A mathematical explanation to the corrections is made by Xie and Wass [8], and using this mathematical explanation, the VCCT calculation formulas of kinking crack are obtained. In the comprehensive survey, another approach to corrections is illustrated by Krueger [3]; it is not dependent on the hypothesis of the singularity of the stress field at the crack tip. The relationship between node opening displacements at the crap tip before crack growth and after are established by taking into account the shape functions of the elements or approximated by simple linear interpolation.

In the paper, a unit form of corrected formulas of VCCT with different element lengths in front of the crack tip and behind is established to the two dimensional crack problems, and two typical modified equations are represented. The SERR is calculated by the VCCT interface element proposed by Xie [9], and then the SIFs can be obtained. The interface element is implemented by the commercial software ABAQUS™ with the user subroutine UEL [10]. Two classical examples (center crack and slanted crack) are computed, and the accuracy and affectivity of the modified equations presented are validated by the excellent agreement with the analytical results. Compared with the traditional modified equations, the modified equations presented in the paper can get more accurate results with the same mesh. Furthermore, the modified VCCT can be simplily implemented in the engineering analysis of complex structure.

## 2 Modified equations

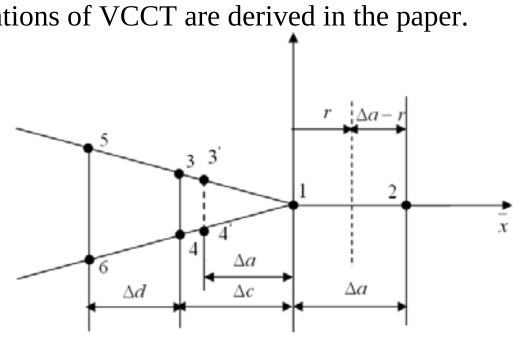
As shown in Fig.1, when the element lengths in front of the crack tip and behind are different, it has two cases: $\Delta a < \Delta c$ and $\Delta c < \Delta a \le \Delta c + \Delta d$ , the case $\Delta a > \Delta c + \Delta d$ is not suggested. The modified equations are presented by Rybichi and Kanninen[7] on the assumption that the stress distribution at the crack tip is same before and after the crack growth:

$$G_I = \lim_{\Delta a \to 0} \frac{\overline{F}_{y1}\Delta\overline{v}_{34}}{2D\sqrt{\Delta a \Delta c}}; G_{II} = \lim_{\Delta a \to 0} \frac{\overline{F}_{x1}\Delta\overline{u}_{34}}{2D\sqrt{\Delta a \Delta c}} \tag{1}$$

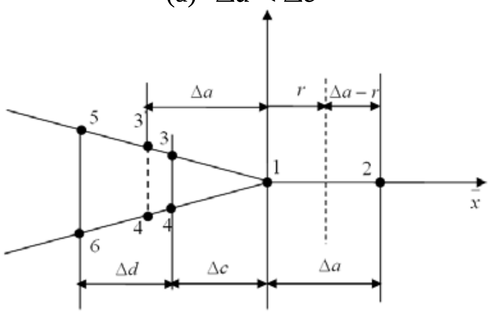
The quantitative relation of the opening displacements at the nodes behind the crack tip before and after the crack growth is establish by Krueger[3], then another kind of modified equations are obtained:

$$G_I = \lim_{\Delta a \to 0} \frac{\overline{F}_{y1}\Delta\overline{v}_{34}}{2D\Delta c}; G_{II} = \lim_{\Delta a \to 0} \frac{\overline{F}_{x1}\Delta\overline{u}_{34}}{2D\Delta c} \tag{2}$$

The two cases of different mesh at the crack tip are not considered in the above two kinds of modified equations, the element length $\Delta a$ in front of the crack tip is not involved in the modified equations given by Krueger. Take the mode I as example, the modified equations of VCCT are derived in the paper.




(a) $\Delta a < \Delta c$




(b) $\Delta c < \Delta a \le \Delta c + \Delta d$

Fig.1. the FE model at the crack tip in local coordinates

Similar to VCCT equations with the same element length in front of the crack tip and behind, the general VCCT equations with different element length in front of the crack tip and behind are as:

$$G_I = \lim_{\Delta a \to 0} \frac{\overline{F}_{y1}\Delta\overline{v}_{3'4'}}{2D\Delta a}; G_{II} = \lim_{\Delta a \to 0} \frac{\overline{F}_{x1}\Delta\overline{u}_{3'4'}}{2D\Delta a} \tag{3}$$

where $\Delta\overline{v}_{3'4'}$ is the opening displacement of nodes behind the crack tip with the distance $\Delta a$.

The element length in front of the crack tip is the virtual extension with a given FE mesh, according to the mathematical explanation presented by Raju[11], the nodal forces at the crack tip are invariable.

$$\int_0^{\Delta a} \overline{\sigma}_{yy}^{(1)}(\overline{x}) \Delta\overline{v}^{(2)}(\overline{x}) d\overline{x} = \overline{F}_{y1}^{(1)} \Delta\overline{v}_{1,1'}^{(2)} \tag{4}$$

Therefore, for the VCCT with different element length in front of the crack tip and behind, the most important work is to calculate $\Delta\overline{v}_{3'4'}$ with the opening displacements supplied by finite element analysis (FEA).

In general, there are three methods to calculate $\Delta\overline{v}_{3'4'}$, the first method to get $\Delta\overline{v}_{3'4'}$ is using $\Delta\overline{v}_{34}$ by linearly interpolating:

$$\Delta\overline{v}_{3'4'} = \frac{\Delta a}{\Delta c} \Delta\overline{v}_{34} \tag{5}$$

The modified equations of VCCT derived by the one-point interpolation method are the same as that given by Krueger. The second method is based on the basic formula $v = B\sqrt{r}$ of the linear elastic fracture mechanics, and then the following relation can be obtained:

$$\Delta\overline{v}_{3'4'} = \sqrt{\frac{\Delta a}{\Delta c}} \Delta\overline{v}_{34} \tag{6}$$

The modified equations of VCCT derived by the basic formula method are the same as that given by Rybichi and Kanninen. The third method to get $\Delta\overline{v}_{3'4'}$ is using $\Delta\overline{v}_{34}$ and $\Delta\overline{v}_{56}$ by linearly interpolating:

$$\Delta\overline{v}_{3'4'} = \frac{\Delta a - \Delta c}{\Delta d} \left( \Delta v_{56} - \Delta v_{34} \right) + \Delta\overline{v}_{34} \tag{7}$$

Then the corresponding modified equations of VCCT are:

$$G_I = \lim_{\Delta a \to 0} \frac{\overline{F}_{y1}}{2D\Delta a} \left[ \frac{\Delta a - \Delta c}{\Delta d} \left( \Delta v_{56} - \Delta v_{34} \right) + \Delta\overline{v}_{34} \right] \tag{8}$$

The method is called two-point interpolation method. The difference between the two cases of different element lengths in front of the crack tip and behind is not considered in the above three methods. In general, for the case $\Delta a < \Delta c$, the result of equation (5) is lower, and the results of equations (6) and (7) are higher; it is on the contrary for the case $\Delta c < \Delta a \le \Delta c + \Delta d$.

Therefore, the mean form of the two methods is used to calculate $\Delta\overline{v}_{3'4'}$ in the paper. By averaging the equations (5) and (6), we get:

$$\Delta\bar{v}_{3'4'} = \frac{1}{2}\left(\frac{\Delta a}{\Delta c} + \sqrt{\frac{\Delta a}{\Delta c}}\right)\Delta\bar{v}_{34} \tag{9}$$

Then the corresponding modified equations of VCCT (called modified equations one) are as follows:

$$G_I = \lim_{\Delta a \to 0} \frac{\bar{F}_{y1}\Delta\bar{v}_{34}}{4D\Delta a}\left(\frac{\Delta a}{\Delta c} + \sqrt{\frac{\Delta a}{\Delta c}}\right) \tag{10}$$

By averaging the equations (5) and (7), we get:

$$\Delta\bar{v}_{3'4'} = \frac{1}{2}\left[\frac{\Delta a}{\Delta c}\Delta\bar{v}_{34} + \frac{\Delta a - \Delta c}{\Delta d}\left(\Delta v_{56} - \Delta v_{34}\right) + \Delta\bar{v}_{34}\right] \tag{11}$$

Then the corresponding modified equations of VCCT (called modified equations two) are as follows:

$$G_I = \lim_{\Delta a \to 0} \frac{\bar{F}_{y1}}{4D\Delta a}\left[\frac{\Delta a}{\Delta c}\Delta\bar{v}_{34} + \frac{\Delta a - \Delta c}{\Delta d}\left(\Delta v_{56} - \Delta v_{34}\right) + \Delta\bar{v}_{34}\right] \tag{12}$$

## 3 The Interface Crack Element

In order to obtain the fracture parameters such as SERRs during the process of FEA, the interface crack element [8] is used around the crack tip in the paper, as is shown in Fig. 2. A simple illustration is conducted in the following.

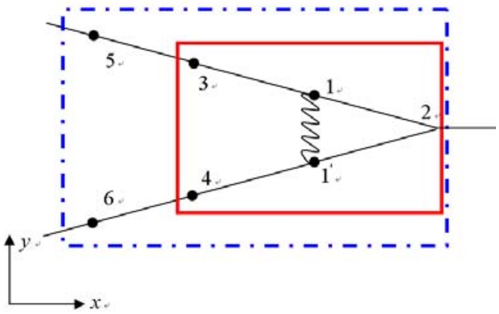



Fig. 2. Definition of interface crack element and its node numbering

In the interface crack element, nodes 1 and $1'$ are located at the crack tip, node 2 in the front of the crack and nodes 3-6 and et al behind the crack tip. The coordinates are in coincidence between nodes 1 and $1'$, 3 and 4, 5 and 6 at the start of FEA.

A very stiff spring is placed between nodes 1 and $1'$ to calculate the nodal forces at the crack tip, it has stiffness $k_x$ in $x$ direction and $k_y$ in $y$ direction. The values of $k_x$ and $k_y$ should be set large enough compared to the stiffness value of the body material to assure the closure at the crack tip if without crack growth.

The introduction of nodes 2-6 is aimed to extract relative information such as nodal displacements from the results of FEA, then the open displacement behind the crack tip and the virtual extended length can be evaluated. These nodes have no contribution to the element stiffness matrix actually, therefore, they are called “dummy nodes” and the interface crack element is also named dummy interface element. The number of dummy nodes is up to the need of calculation of fracture parameters including SERRs et al.

The interface crack element is implemented with the user subroutine UEL of software ABAQUS™. The Jacobian matrix $[K]$ and residual vector $\{R\}$ must be provided in UEL. For the interface crack element presented, without regard to the tractions applied in the crack surfaces (if exist, consider them in the body element), the expressions of $[K]$ and $\{R\}$ are as following

$$[K] = \begin{bmatrix} k_x & 0 & -k_x & 0 \\ 0 & k_y & 0 & -k_y \\ -k_x & 0 & k_x & 0 \\ 0 & -k_y & 0 & k_y \end{bmatrix}, \{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_{1'} \\ v_{1'} \end{Bmatrix},$$

$$\{R\} = -[K]\{u\} \tag{13}$$

It is worthy noted: the information supplied by software ABAQUS™ is based on the global coordinate system and it need to be transformed into that of the local coordinate system at the crack tip when the SERRs are calculated.

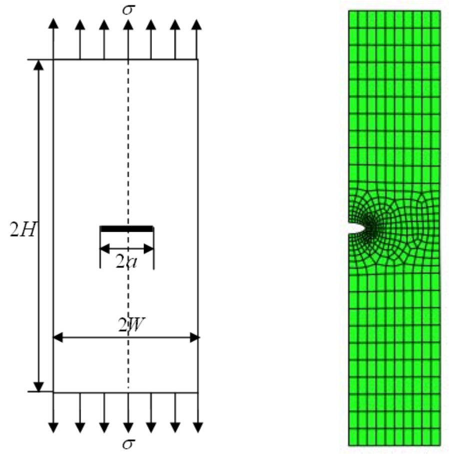
## 4 Numerical Results

In order to vertify the accuracy of the present modified equations of VCCT, two classical examples are evaluated.

### 4.1 Fracture problem with center crack

As is shown in Fig.3(a), the heigth, width and thickness are $2H$ , $2W$ and $D$. It is a classical I-mode fracture problem, and the analytic result of the infinite plate with a center crack is[12]:

$$K_I = \sigma\sqrt{\pi a}\sqrt{\sec\frac{\pi a}{2W}}\left[1 - 0.025\left(\frac{a}{W}\right)^2 + 0.06\left(\frac{a}{W}\right)^4\right] \tag{14}$$



(a) Geometrical model (b) FE model

Fig.3 Center crack model

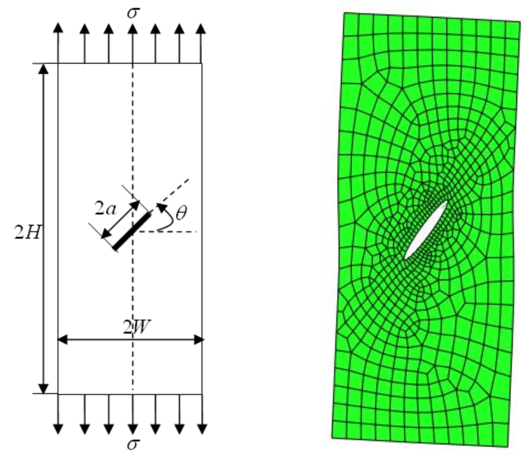
In order to compare with the analytic result, set $a/W$ =0.2, $H/W=2$ , $W=100$ mm, $D=1$ mm, elastic module $E=200$ GPa, Poisson's ration $\nu=0.25$. Based on the symmetry, half of the model is analyzed, the element length in front of the crack tip is set $\Delta a=1$ mm, when meshing the structure, the bias ratio is 2.0, and the part near the crack tip is dense. Then by setting the different seed number on the crack line, we can obtain different $\Delta c/\Delta a$. The free mesh method is selected in ABAQUS[TM], the element is four-node plane stress element CPS4, the FE mesh is shown in Fig.3 (b). It is noted that the quality of mesh in front of the crack tip and behind has great influence on the results, and sometimes the mesh in front of the crack tip and behind is not perfect when the free mesh method is selected, therefore, it needs to modify the coordinates of one or more nodes.

Table 1 The relative errors of SIFs

| $\Delta c/\Delta a$ | one-point interpolation method | basic formula method | modified equations one | two-point interpolation method | modified equations two |
|---|---|---|---|---|---|
| 0.447 | 25.958 | -15.825 | 4.559 | -6.761 | 9.374 |
| 0.513 | 17.797 | -15.657 | 1.070 | -6.929 | 5.434 |
| 0.554 | 16.716 | -13.158 | 1.779 | -8.123 | 4.297 |
| 0.602 | 11.425 | -13.572 | -1.074 | -7.695 | 1.865 |
| 0.659 | 11.619 | -9.401 | 1.109 | -7.453 | 2.080 |
| 0.728 | 7.015 | -8.691 | -0.838 | -7.095 | -0.040 |
| 0.864 | 2.486 | -4.730 | -1.122 | -4.518 | -1.016 |
| 1.000 | -2.820 | -2.820 | -2.820 | -2.820 | -2.820 |
| 1.063 | -4.800 | -1.854 | -3.327 | -1.587 | -3.194 |
| 1.255 | -9.700 | 1.175 | -4.261 | 2.554 | -3.572 |
| 1.332 | -9.829 | 4.097 | -2.866 | 7.434 | -1.197 |
| 1.533 | -13.694 | 6.849 | -3.422 | 11.811 | -0.941 |
| 1.967 | -20.123 | 12.039 | -4.042 | 24.081 | 1.980 |
| 2.292 | -23.827 | 15.328 | -4.249 | 30.902 | 3.538 |

The relative errors of SIFs ($e_r=\left(K_{I0}-K_I\right)/K_{I0}$) obtained by different method are listed in Table 1, where $K_{I0}$ is the result of equation (14), $K_I$ is the result of different method. It is clearly seen: when $\Delta c/\Delta a$ is not equal to 1.0, the errors of one-point interpolation method, basic formula method and two-point interpolation method are very high, and the errors of modified equations one and modified equations two are low, the error of modified equations one is within 5%, the error of modified equations two is also within 5% when $0.554<\Delta c/\Delta a<2.292$; The errors of all the five methods increase greatly as the mismatch level of the element lengths in front of the crack tip and behind increases.

### 4.2 Fracture problem with slanted crack



(a) Geometrical model (b) FE model

Fig.4 Slanted crack model

The geometrical model of the slanted crack plate is shown in Fig.4 (a). In order to compare with the results in Reference [12], set $a/W = 0.4$ , $h/W = 2$ , $\theta = 45^0$ , $W = 2.5$ m, $D = 1$ m,elastic module $E = 200$ GPa, Poisson's ration $\nu = 0.3$ , $\sigma = 100$ MPa, the element length in front of the crack tip $\Delta a = 0.5$ m. When meshing the structure, devide the crack line at the middle point and set the same seed number to each part. For the case $\Delta c/\Delta a \geq 1.0$ , we can obtain different $\Delta c/\Delta a$ by setting the different seed number; for the case $\Delta c/\Delta a < 1.0$ , the bias ratio of the seeds is 2.0, and the part near the crack tip is dense, then set the different seed number to obtain different $\Delta c/\Delta a$ . The free mesh method is selected in ABAQUS™, the element is four-node plane strain element CPE4, the FE mesh is shown in Fig.4 (b).

Table 2 The relative errors of mode-I SIFs

| $\Delta c/\Delta a$ | one-point interpolation method | basic formula method | modified equations one | two-point interpolation method | modified equations two |
|---|---|---|---|---|---|
| 0.494 | 26.459 | -11.084 | 7.687 | -9.785 | 8.337 |
| 0.532 | 23.970 | -9.549 | 7.210 | -10.523 | 6.723 |
| 0.577 | 20.518 | -8.483 | 6.017 | -10.123 | 5.197 |
| 0.629 | 17.707 | -6.652 | 5.528 | -9.480 | 4.114 |
| 0.659 | 15.320 | -6.396 | 4.462 | -8.950 | 3.185 |
| 0.692 | 13.233 | -5.825 | 3.704 | -8.580 | 2.327 |
| 0.833 | 5.327 | -3.850 | 0.738 | -5.937 | -0.305 |
| 1.000 | -1.531 | -1.531 | -1.531 | -1.531 | -1.531 |
| 1.250 | -8.768 | 2.000 | -3.384 | 6.442 | -1.163 |
| 1.429 | -10.741 | 6.685 | -2.028 | 13.406 | 1.333 |
| 1.538 | -15.026 | 5.397 | -4.815 | 14.192 | -0.417 |
| 1.667 | -17.876 | 6.022 | -5.927 | 18.422 | 0.273 |
| 1.818 | -20.231 | 7.560 | -6.335 | 23.012 | 1.391 |
| 2.000 | -21.594 | 10.882 | -5.356 | 31.134 | 4.770 |
| 2.222 | -25.248 | 11.434 | -6.907 | 37.147 | 5.950 |
| 2.500 | -26.674 | 15.939 | -5.367 | 48.390 | 10.858 |

Table 3 The relative errors of mode-II SIFs

| $\Delta c/\Delta a$ | one-point interpolation method | basic formula method | modified equations one | two-point interpolation method | modified equations two |
|---|---|---|---|---|---|
| 0.494 | 22.113 | -14.140 | 3.987 | -5.119 | 8.497 |
| 0.532 | 20.110 | -12.366 | 3.872 | -5.830 | 7.140 |
| 0.577 | 15.635 | -12.192 | 1.721 | -6.167 | 4.734 |
| 0.629 | 12.733 | -10.596 | 1.068 | -6.734 | 2.999 |
| 0.659 | 11.645 | -9.380 | 1.132 | -6.069 | 2.788 |
| 0.692 | 9.504 | -8.927 | 0.288 | -6.394 | 1.555 |
| 0.833 | 3.909 | -5.144 | -0.617 | -4.807 | -0.449 |
| 1.000 | -2.128 | -2.128 | -2.128 | -2.128 | -2.128 |
| 1.250 | -8.987 | 1.755 | -3.616 | 3.055 | -2.966 |
| 1.429 | -10.355 | 7.146 | -1.604 | 7.315 | -1.520 |
| 1.538 | -13.975 | 6.701 | -3.637 | 10.184 | -1.895 |
| 1.667 | -15.901 | 8.572 | -3.664 | 14.700 | -0.600 |
| 1.818 | -17.411 | 11.363 | -3.024 | 19.464 | 1.027 |
| 2.000 | -20.536 | 12.379 | -4.078 | 23.048 | 1.256 |
| 2.222 | -22.862 | 14.991 | -3.935 | 31.560 | 4.349 |
| 2.500 | -23.550 | 20.878 | -1.336 | 40.206 | 8.328 |

The relative errors of mode-I and mode II SIFs ( $e_r = (K_{i0} - K_i)/K_{i0}$ , $i = I, II$ ) obtained by different method are listed in Table 2 and Table 3, where $K_{i0}$ is the analytic result, $K_{I0} = 1.0137E8$ ( $Pa \cdot \sqrt{m}$ ), $K_{II0} = 0.9376E8$ ( $Pa \cdot \sqrt{m}$ ), $K_i$ is the result of different method. It is clearly seen: the errors of one-point interpolation method, basic formula method and two-point interpolation method are very high, and the errors of modified equations one and modified equations two are low, the error of modified equations one is within 4%, the error of modified equations two is high when the mismatch level of the element lengths in front of the crack tip and behind is high.

## Conclusions

The VCCT with different element lengths in front of the crack tip and behind is investigated in the paper, and the following conclusions are made:

(1) The modified equations of VCCT with different element lengths in front of the crack tip and behind are given out in the paper. when the ratio of the element lengths in front of the crack tip and behind $\Delta c / \Delta a$ is within 0.5~2.0, the modified equations of VCCT presented in the paper lead to accurate results.

(2) The two kinds of modified equations proposed in the paper are suited for the two cases( $\Delta a < \Delta c$ and $\Delta c < \Delta a \le \Delta c + \Delta d$ ), and one-point interpolation method, basic formula method and two-point interpolation method are not applied when the mismatch level of the element lengths in front of the crack tip and behind is high.

(3) It is best to assure the ratio of the element lengths in front of the crack tip and behind near to 1.0 in the complex problems such as mixed-mode fracture problem. The modified equations in the paper can give more accurate results in this way.

(4) The fracture parameters can be easily calculated during the process of FEA with interface crack element and no extra convergence issues are meet. The interface crack element can also be used with other numerical methods.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## References


[1] B D Davidson, S J Gharibian and L J Yu. Evaluation of energy release rate-based approaches for predicting delamination growth in laminated composites. International Journal of Fracture, 105, 343-365, 2000.

[2] TE Tay. Characterization and analysis of delamination fracture in composites: A review of development from 1990 to 2001. Applied Mechanic Review, V56, 1, 1-32, 2003.

[3] R Krueger. Virtual crack closure technique: History, approach, and applications. Applied Mechanic Review, V57, 2, 109-143, 2004.

[4] L Banks-Sills. Application of the finite element method to prediction of onset of delamination growth. Applied Mechanic Review, V44, 10, 447-461, 1991.

[5] L Banks-Sills. Update: Application of the finite element method to linear elastic fracture mechanics. Applied Mechanic Review, 63, 1-17, 2010.

[6] A Miravete and MA Jimenez. Application of the finite element method to prediction of onset of delamination growth. Applied Mechanic Review, V55, 2, 89-105, 2002.

[7] EF Rybicki and MF Kanninen. A finite element calculation of stress intensity factors by a modified crack closure integral. Engineering Fracture Mechanics, 9, 931-938, 1977.

[8] D Xie, A M Wass and Shahwan, et al. Computation of strain energy release rate for kinking cracks based on virtual crack closure technique. CMES: Computer Modeling in Engineering & Sciences, 6, 515-524, 2004.

[9] D Xie and S B Biggers Jr. Progressive crack growth analysis using interface element based on the virtual crack closure technique. Finite Element in Analysis and Design, 42, 977-984, 2006.

[10] HKS, inc. ABAQUS User Manual. Version 6.7. HKS, Pawtucket.

[11] IS Raju. Calculation of strain-energy release rates with high-order and singular finite-elements. Engineering Fracture Mechanics, 28, 251-274, 1987.

[12] D Xie, Qian Q, C A Li Edit. The numerical methods and engineering application in fracture mechanics. Science Press. 4-7, 91-93, 2009 (In Chinese)