FIBER COMPRESSIVE FAILURE CRITERION AS SHEAR BAND MODE BIFURCATION CONDITION

T. Nadabe1*, N. Takeda1

1 Department of Advanced Energy, The University of Tokyo, Tokyo, Japan

* Corresponding author (nadabe@smart.k.u-tokyo.ac.jp)

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1 Introduction

Fiber kinking failure shown in Fig. 1(a) is commonly observed in compressive failure of fiber reinforced composite materials. It is a band of localized shear deformation, and in many cases it is considered as a result of material instability [1]. The localization phenomena is observed in various kinds of materials such as Lüders bands in metals, necking in polymers, faults in rock mass and shear bands in soils, and those have been investigated as shear band mode bifurcation, in which material instability due to inelastic behavior of the material plays an important role [2]. Fiber kinking in composite materials is also considered to be categorized in those localization phenomena. For the localization phenomena, Borja [2] investigated shear band mode bifurcation for general elasto-plastic solids in the frame of finite deformation theory. In this study, we follow the bifurcation theory given by Borja [2], and obtain the localization condition in fiber kinking failure as the shear band mode bifurcation condition. The obtained condition is treated as failure criterion of fiber compressive failure, and is applied to progressive failure analysis in composite materials.

2 Localization Condition in Composite Materials as Shear Band Mode Bifurcation Condition

Following the bifurcation theory shown by Borja [2], we obtain the localization condition in composite materials, which represents the behavior of failure in fiber kinking failure. Here, Updated Lagrangian formulation is applied. First, the rate form boundary value problem at present configuration is defined as follows,

\[ \dot{\Pi} \cdot n = i \quad \text{on} \ S_i \]  \hspace{1cm} (2)

\[ v = v_s \quad \text{on} \ S_u \]  \hspace{1cm} (3)

where \( \Pi \) is nominal stress, \( \rho \) is density, \( b \) is body force, \( n \) is unit normal on boundary, \( t \) is surface force, \( v \) is velocity, and material derivative of any tensor \( \chi \) is represented as \( \dot{\chi} = \nabla \chi \). \( V \) is body and \( tS, uS \) is boundary surfaces. Then stress vectors \( t_A, t_B \) on the upper and lower surfaces \( S_A, S_B \) of shear band are represented using nominal stress as follows,

\[ \dot{\Pi}_A \cdot n_A = i_A \quad \text{in} \ S_A \]  \hspace{1cm} (4)

\[ \dot{\Pi}_B \cdot n_B = i_B \quad \text{on} \ S_B \]  \hspace{1cm} (5)

At the moment of localization onset, \( n \) and \( t \) are continuous across the shear band, then

\[ (\dot{\Pi}_A - \dot{\Pi}_B) \cdot n = [[\Pi]] \cdot n = 0 \]  \hspace{1cm} (6)

where \( n \) is unit normal on initiating shear band, \( [[ \cdot ]] \) represents the amount of discontinuity across the shear band. In addition, the constitutive equation at finite deformation is represented as follows,

\[ \Pi^T = A : L \]  \hspace{1cm} (7)

where \( A \) is tangential moduli tensor at finite deformation and \( L \) is velocity gradient. From Eqs. (6-7),

\[ [[A : L]] \cdot n = 0 \]  \hspace{1cm} (8)

The discontinuity of velocity gradient across the shear band is represented as [2],

\[ [[L]] = ([[v]] \otimes n)/h \]  \hspace{1cm} (9)

where \( h \) is the width of shear band. From Eqs. (8-9), when tangential moduli tensor \( A \) is continuous,
Taking Eq. (10),
\[ a \cdot [v] = 0, \quad a_i = n_i A_{ij} n_j \] (11)

Tensor \( a_i \) is called acoustic tensor. The condition for the existence of non zero discontinuous velocity field \([v]\) is represented as follows,
\[ \det a_{ij} = 0 \] (12)

Thus the shear band mode bifurcation condition is obtained from the requirement of continuity of the nominal traction vector on the potential shear band, and the loss condition of the uniqueness of the solution for strain field. The localization condition in fiber kinking is represented by the bifurcation condition in Eq. (12).

The predicted results of shear band mode bifurcation largely depend on the constitutive description of homogeneous deformation [2], since the localization of deformation is closely related with the inelastic behavior of the materials. In this study, as the constitutive description of fiber reinforced composite materials, we apply the nonlinear deformation theory given by Tohgo et al. [3], in which the relation between stress rate and strain rate is represented as follows,
\[ \varepsilon = \sigma \varepsilon_d \] (13)

where \( f_{comp} \), \( m \), and \( S \) are tangential moduli tensor of composites, fibers and matrix, respectively. \( f_{V} \) is fiber volume fraction. To obtain tangential moduli tensor of matrix, the evaluation for equivalent stress of matrix is necessary. The following relation is applied to evaluate the equivalent stress of matrix from the applied stress in composite materials.
\[ d\sigma = C_{comp} d\varepsilon \]
\[ C_{comp} = C_{m} \left\{ \left[1 - V_f\right] (C_f - C_m)S + C_m \right\}^{-1} K \] (14)
where \( C_{comp}, C_f, \) and \( C_m \) are tangential moduli tensor of composites, fibers and matrix, respectively. \( S \) is Eshelby tensor and \( V_f \) is fiber volume fraction. To obtain tangential moduli tensor of matrix, the evaluation for equivalent stress of matrix is necessary. The following relation is applied to evaluate the equivalent stress of matrix from the applied stress in composite materials.
\[ d\sigma_m = C_m (S - I)K^{-1} \left\{ (C_f - C_m)S + C_m \right\} : (S - I)^{-1} C_m^{-1} d\sigma \] (15)

In addition, the fiber direction of composite materials changes during the deformation of the material. The constitutive relation is anisotropic, and it is required to be always defined with the local fiber direction. In finite deformation analysis, this is the problem of objective stress rate. In order to follow the local fiber direction which changes with the material deformation, stress rate such as Oldroyd rate should be the most appropriate, and the corotational rate such as Jaumann rate has difficulty, since rigid body rotation is not the same as the rotation of local fiber direction because of the material deformation. Additionally, since compressive failure is largely affected by local fiber direction, analysis result of compressive failure depends on objective stress rate. In the following, the Oldroyd rate of Kirchhoff stress is applied as objective stress rate. Then the constitutive relation of composite materials is represented as follows,
\[ \tau = \sigma_{comp} : D \] (16)
where \( \tau \) is Oldroyd rate of Kirchhoff stress and \( D \) is deformation rate. In the Updated Lagrangian formulation, material derivative of nominal stress \( \Pi \) is represented as follows,
\[ \Pi^T = \tau + L \cdot \sigma \] (17)
where \( \sigma \) is Cauchy stress. From Eqs. (7, 15-16),
\[ A : L = C_{comp} : D + L \cdot \sigma \] (18)

In this equation, second term of RHS is written as follows,
\[ l_i \sigma_{ij} = \sigma_{ij} \delta_{ij} l_{ij} \] (19)

Then Eq. (16) is represented as,
\[ A_{ijkl} l_{ij} = (c_{ijkl} + \sigma_{ij} \delta_{ij}) l_{ij} \] (20)
where \( d_{ij} = \text{sym}(l_{ij}) \) is applied. Tensor \( A_{ijkl} \) is represented as,
\[ A_{ijkl} = c_{ijkl} + \sigma_{ij} \delta_{ij} \] (21)

Therefore the acoustic tensor \( a_{ij} \) in Eq. (11) is represented as follows,
\[ a_{ij} = c_{ij} n_i n_j + \sigma_{ij} n_i n_j \delta_{ij} \] (22)

In the following section, the explicit expressions for compressive strength are obtained from the bifurcation condition, and they are compared with
the previous models for fiber compressive failure.

3 Explicit expressions for compressive strength

Here we assume the plane strain condition, and consider two dimensional problem. In two dimensional problem, unit normal \( n_i \) on band surface in Eq. (21) is represented as follows,

\[
\{n_i, n_z\} = \{\cos \beta, \sin \beta\}
\]

(22)

where \( \beta \) is angle between the loading direction and unit normal on the band and it is equivalent to the kink band angle. Applying Eq. (22) on Eq. (21), the explicit expression for acoustic tensor \( a_{ij} \) is obtained.

\[
a_{11} = (c_{11} + \sigma_{11})k\beta^2 + 2(c_{16} + \sigma_{16})k\beta\sigma + (c_{16} + \sigma_{22})\beta^2
\]

\[
a_{22} = (c_{66} + \sigma_{16})k\beta^2 + 2(c_{26} + \sigma_{26})k\beta\sigma + (c_{26} + \sigma_{22})\beta^2
\]

\[
a_{12} = c_{66}k\beta^2 + (c_{16} + c_{66})k\beta\sigma + c_{16}\beta^2
\]

\[
a_{21} = c_{66}k\beta^2 + (c_{16} + c_{66})k\beta\sigma + c_{16}\beta^2
\]

(23)

where \( c \) is \( \cos \beta \), \( s \) is \( \sin \beta \), tangential moduli is represented in second rank tensor. Using these equations, the bifurcation condition in Eq. (12) is represented as,

\[
\det a_{ij} = a_{11}a_{22} - a_{12}a_{21} = 0
\]

(24)

Thus from Eqs. (23-24), the bifurcation condition is determined by tangential moduli \( c_{ij} \) derived from material stiffness, multiaxial stresses \( \sigma_{ij} \) derived from geometrical stiffness and kink band angle \( \beta \) related to orientation of localization.

For many cases in fiber reinforced composites, \( c_{11} \gg c_{ij} \) (\( c_{ij} \neq c_{11} \)). Using this approximation in Eqs. (23-24), \( a_{ij} \) has much higher value and \( a_{22} = 0 \). Then from Eq. (23), the critical compressive stress in fiber direction is expressed as follows,

\[
\sigma_{cr} = -\sigma_{11} = c_{66} + 2(c_{26} + \sigma_{16})\tan \beta + (c_{26} + \sigma_{22})\tan^2 \beta
\]

(25)

This represents the compressive strength of composite materials for fiber kinking failure. Particularly in the case of uniaxial compression and if \( c_{26} = 0 \), the compressive strength is approximately represented as follows,

\[
\sigma_{cr} = G_{LT} + E_T \tan^2 \beta
\]

(26)

where \( G_{LT} \) is shear tangential modulus and \( E_T \) is transverse tangential modulus. Eq. (26) corresponds with the expression given by Budiansky [4]. Meanwhile, when the Jaumann rate of Kirchhoff stress is applied, the corresponding expression to Eq. (25) is as follows,

\[
\sigma_{cr} = -\sigma_{11} = 2c_{66} - \sigma_{22} + 4c_{26}\tan \beta + 2(c_{26} - \sigma_{22})\tan^2 \beta
\]

(27)

and for uniaxial compression, if \( c_{26} = 0 \),

\[
\sigma_{cr} = 2(G_{LT} + E_T \tan^2 \beta)
\]

(28)

The variance between Eqs. (25) and (27) comes from the difference of objective stress rate. Thus the reason for the variance between Eq. (28) and the previous model is due to the difference between the rigid body rotation and the rotation of local fiber direction owing to the material deformation.

Here, let us consider two kinds of two dimensional model for composite materials as shown in Fig. 2. Fig. 2(a) is the model assuming fibers and matrix as plate (model 1), and Fig. 2(b) is the model where fibers are cylinder solids and matrix surrounds fibers, and considering two dimensional problem of the composites in macroscale (model 2). Due to the difference of types of fibers in model 1 and 2, the form of Eshelby tensor is different. Applying Eshelby tensors for each model in Eq. (13), the expression for compressive strength is obtained.

When \( \beta = 0 \) and \( G_M >> G_n \), compressive strength is represented as follows,

\[
\text{Model 1 } \quad \sigma_{cr} = \frac{G_M}{1 - V_f}
\]

(29)

\[
\text{Model 2 } \quad \sigma_{cr} = \frac{1 + V_f}{1 - V_f} G_m
\]

(30)

where \( G_m \) is matrix shear modulus. Eq. (29) agrees with the expression for fiber microbuckling stress at shear mode given by Rosen [1].

In addition, the degradation of tangential modulus of matrix after yield is trigger for the localization of deformation. Thus the bifurcation condition is closely related with the degradation condition of tangential modulus in matrix. The another expression for compressive strength is obtained from the yield condition of matrix at specific degradation point of tangential modulus, Eq. (14), and the equilibrium condition of applied stress in between
misalignment coordinate system and coordinate system associated with global fiber direction. When $G_f \gg G_n$, the compressive strength is represented as follows,

$$\sigma_{cr} = -\sigma_{11} \left(1 + V_f\right) - \frac{\tau_{12}}{\phi} - \sigma_{22}$$

(31)

In the case of uniaxial compression, it is written as,

$$\sigma_{cr} = \frac{\tau_{12}(1 + V_f)}{\phi}$$

(32)

When $V_f$ is constant, this expression also corresponds with the model given by Budiansky et al. [5]. In addition, the relation between compressive strength and fiber volume fraction $V_f$ is approximately represented as linear relation. Thus the explicit expressions for compressive strength obtained from the shear band mode bifurcation condition corresponds well with the previous models for fiber compressive failure, when the Oldroyd rate of Kirchhoff stress as embedded rate is used. Here, when the Oldroyd rate of Cauchy stress is used, similar correspondence was found. In the following section, the compressive failure is numerically analyzed using the bifurcation condition.

4 Numerical Results on Compressive strength

To analyze compressive failure numerically, incremental analysis is conducted. Stress is applied incrementally, and in each increment, the procedure to analyze bifurcation is as follows,

1. Transform stress in coordinate system associated with the fiber aligned direction in order to take into account the effect of fiber misalignment.
2. Evaluate mean stress in matrix using Eq. (14).
3. Evaluate equivalent stress in matrix.
5. Calculate tangential moduli tensor of composite in Eq. (13).
6. Calculate acoustic tensor $a_{ij}$ in Eq. (21).
7. Check bifurcation condition in Eq. (12).

When the bifurcation condition is satisfied, failure is assumed to initiate. In this analysis, material property for CFRP T700S/2592 (Toray Industries Inc.) [6] is applied. For strain hardening curve of matrix, two kinds of hardening curves M and N shown in Fig. 3 are used and the results are compared. In addition, to investigate the influence for the differences of objective stress rates, the different types of objective stress rates as shown in Table 1 are applied and the results are compared. Furthermore, for matrix failure, failure criteria presented by Pinho et al. [7] is applied. Figs. 4(a) and (b) shows the compressive strength under multiaxial stress states for hardening M and N, respectively. In the results of hardening N, the difference for each objective stress rate is relatively large. It depends on the hardening behavior of matrix. In the following, hardening N is used for the analysis. In addition, the effects of multiaxial stresses on compressive strength are closely related with matrix yield behavior under multiaxial stress states.

Fig. 5 shows the effects of material property on compressive strength. The dependency of compressive strength for material property is well simulated. Thus the characteristics of compressive strength is well reproduced in numerical analysis using shear band mode bifurcation condition.

5 Implementation in Progressive Failure Analysis

Here we assume the shear band mode bifurcation condition as failure criterion of fiber compressive failure, and apply to progressive failure analysis. The bearing failure in CFRP bolted joints is analyzed using the progressive failure analysis. In this progressive failure analysis, for each increment and finite element, the bifurcation condition is checked using the numerical procedure described in Section 4. If the bifurcation condition is satisfied, then the localization is assumed to occur at that area and failure is assumed to initiate. Similar to the previous section, for matrix failure, failure criteria by Pinho et al. [7] is applied. After failure initiates, constitutive damage model in Ref. [9] is applied. As the finite element model of bolted joints, the model in Ref. [8, 9] is used. The model is shown in Fig. 6. Finite element code Abaqus is used for the analysis. In bifurcation analysis, both elastic and elasto-plastic analysis is examined.

Fig. 7 shows the simulated load-displacement curve in bearing failure. The simulated first peak load for elasto-plastic case largely agrees with the
experimental result [8,9]. Before fiber kinking failure initiates, matrix yield occurs at the potential failure initiation point. Therefore, there is a possibility to know the initiation of fiber kinking failure beforehand, by detecting the local matrix yield in the potential areas. Fig. 8 shows the simulated fiber kinking area in 0º ply with experimental result [8,9]. The simulated fiber kinking area in 0º ply also agrees with the experimental result.

6 Conclusions

The localization condition in fiber kinking failure is represented as shear band mode bifurcation condition. The explicit expressions for compressive strength agree with the previous models for compressive failure, and in numerical analysis, the characteristics for compressive strength is reproduced. The bifurcation condition is applied as fiber compressive failure criterion to progressive failure analysis, and bearing failure in CFRP bolted joints is analyzed. The first peak load and damage area in 0º ply agree with the experimental results.

Acknowledgements

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References

[6] TORAYCA technical reference, Toray Industries Inc..
Fig. 4 Compressive strength under multiaxial stress states for different objective stress rates. a-d in these figures correspond with the ones in Table 1.

Fig. 5 Effects of material property on compressive strength for hardening N. a-d in this figure correspond with the one in Table 1.

Table 1 Objective stress rate examined in this study

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Fig. 6 Finite element model of bolted joints [8,9]. Stacking sequence of CFRP laminate is $[45/0/-45/90]$. 

Fig. 7 Simulated load-displacement curve in bearing failure.

Fig. 8 Fiber kinking area in $0^\circ$ ply in bearing failure.