EFFECT OF SPATIAL RANDOM POISSN'S RATIO ON THE IN-PLANE BEHAVIOR OF COMPOSITE PLATES

H. Noh[1]*, D. Seo[2] , D. Kim[3]
[1, 2,3] Department of Civil and Environmental Engineering, Sejong University, Seoul, Korea
* Corresponding author (cpebach@sejong.ac.kr)

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1 General Introduction

As the understandings of materials and structures are accumulated, it becomes clear that the randomness in their natural state is inevitable. According to the research results in the literature[1,2], the random structural parameters, whether they are material parameters or geometrical ones, affects the behavior of the structures, and the random behavior is of importance from the perspective that the degree of randomness in the response is large enough to be ignored. In this study, we investigate into the effect of random Poisson's ratio[3] on the uncertain responses of composite materials especially the in-plane behavior. Among the various stochastic FE(finite element) analysis schemes, the weighted integral scheme shows more accurate results [4] and in this paper the proposed formulation has its base in the weighted integral method. Regarding the probabilistic distribution of the random parameter under consideration, we assume they are Gaussian parameter, which is acceptable if the coefficient of variation is as small as around 0.1[1,5].

2 Random Poisson's Ratio

2.1 Expression of Randomness

Adopting the general mathematical expression, the random Poisson's ratio can be written as:

$$\nu(\mathbf{x}) = \bar{\nu}\left[1 + f_\nu\right] \tag{1}$$

where the symbol with over-bar denotes the mean of the parameter and f_ν is the stochastic field function of the random parameter. With Eq. (1) it is obvious that the mean of the stochastic field function is vanishing: $\varepsilon\left[f_\nu\right] = 0.0$. Eq. (1) also says that the random parameter can be divided into two parts: mean and deviator. In order for the parameter to have physical meaning, the stochastic field function needs to be in the range $-1+\varepsilon_f < f_\nu(\mathbf{x}) < 1-\varepsilon_f$, where $0.0 < \varepsilon_f < 1.0$.

2.2 Reciprocal Relation in Poisson's ratio

In case of composite materials which consist of resin and fiber, there is the reciprocal relation as given by the following equation between Poisson's ratio and elastic modulus:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{2}$$

As a consequence, the random Poisson's ratio in one direction can be represented by the Poisson's ratio in the other normal direction, which is multiplied by the ratio of elastic moduli in two directions, i.e.,

$$\nu_{ij} = r_E^2 \nu_{ji}, \quad r_E = \sqrt{E_i / E_j} \tag{3}$$

2.3 Coefficients for stress-strain relation

Using the relation in Eq. (3), the coefficients in the stress-strain relation of the composites can be given, for example, as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = \frac{E_1}{1-r_E^2\nu_{12}^2} \tag{4}$$

The other coefficients $Q_{ij}, i = 1,2, \ j = 2$ also can be expressed in an analogous way. From Eq. (4) we can note that coefficients are in the fraction form of $1/(1-x)$, where x is replacing $r_E^2\nu_{12}^2$. Employing Taylor's expansion on $1/(1-x)$ and then substituting $\nu_{12}(\mathbf{x})$ as given in Eq. (1), the expansion results in the following expression

$$Q_{11} = E_1\left[1+\sum_{l=0}^{\infty}\left\{\sum_{k=1,(2k\geq l)}^{\infty}\left(r_E\bar{\nu}_{12}\right)^{2k}\ {}_{2k}C_l\right\}f_\nu^l\right] \quad (5)$$

Introducing constants for inner summation in Eq. (5) since they are having only constant quantities, Eq. (5) can be rewritten in a simpler form as

$$Q_{11} = E_1\beta_i f_\nu^i,\ i=1,\cdots,\infty \qquad (6)$$

In fact, the other two coefficients can also be expressed in a similar form. In Eq. (6) the repeated index means summation, and the constants are, for example:

$$\beta_i = \sum_{k=1,(2k\geq i)}^{\infty}\left(r_E\bar{\nu}_{12}\right)^{2k}\ {}_{2k}C_i \qquad (7)$$

Even though the expansion in Eq. (5) goes to infinity, the higher order terms than 4 can be truncated out with only minor errors because the exponent $r_E\bar{\nu}_{12}$ is much less than 1.0. The constants for Q_{12} is a little different from those for Q_{11}, Q_{22}.

2.4 Stress-strain relation in global coordinates

The coefficients Q_{ij} in the previous section are derived in the material coordinate system. Those coefficients can be transformed to lamia(global) coordinate system depending on the stacking angle θ. With c and s implying cosine and sine for the stacking angle, the coefficients in the lamina coordinate can be functions of original c, s and the original coefficients, which can be expressed in a brief form as

$$\bar{Q}_{ij} = F\left(Q_{ij}, c, s\right) \qquad (8)$$

Combining Eqs. (6) and (8), the coefficients $\bar{Q}_{ij}$ are given in the following form, which has a stochastic field function in the increasing order as

$$\bar{Q}_{ij} \cong \tilde{Q}_{ij(l)} f_\nu^l \qquad (9)$$

As expected, the coefficients $\tilde{Q}_{ij(l)}$ s are expressed in terms of elastic modulus, β_i and the stacking angle θ. As a result, the material matrix $\mathbf{Q}$ in the lamina coordinate can be given as follows:

$$\bar{\mathbf{Q}} \cong \tilde{\mathbf{Q}}_{(l)} f_\nu^l \qquad (10)$$

where the sub-matrix $\tilde{\mathbf{Q}}_{(l)}$ s are

$$\tilde{\mathbf{Q}}_{(l)} = \begin{bmatrix} \tilde{Q}_{11(l)} & \tilde{Q}_{12(l)} & \tilde{Q}_{16(l)} \\ & \tilde{Q}_{22(l)} & \tilde{Q}_{26(l)} \\ sym. & & \tilde{Q}_{66(l)} \end{bmatrix} \qquad (11)$$

3. Element Stiffness with Randomness

3.1 Expression for Stress Resultants

Generally accepted matrix notations of the composite materials for stress resultants for extension, bending and coupling terms are A, D and B, and respective terms are established by the summation of the contribution of each layers as

$$\begin{aligned}\left(A_{ij}, B_{ij}, D_{ij}\right) &= \int_{-t/2}^{t/2}\bar{Q}_{ij}\left(1, z, z^2\right)dz \\ &= \frac{1}{p}\sum_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\left(z_{k+1}^p - z_k^p\right)\end{aligned} \qquad (12)$$

Where t denotes the thickness of the composite plate and the thickness of (k)-th layer is given as

$$t^{(k)} = z_{k+1} - z_k \qquad (13)$$

If we use E as a unified notation for the total stress resultant, we can write

$$\mathbf{E} = \mathbf{E}_{\det} + \delta\mathbf{E} \qquad (14)$$

where $E_{\det}$ denotes the deterministic stress resultant matrix, which is different from that of the mean matrix, and δE denotes $E^{(l)} f_\nu^l$. Even though the expanded expression includes infinite number of terms, only up to 4th terms are retained in the computation for the computational applicability assuming the error of truncation is small enough to be ignored.

3.2 Stiffness and Random Variables

Due to the effect randomness in the Poisson's ratio, the stiffness is to be divided into two parts: the mean and the deviator. This division can be seen in the following expression for the stiffness:

$$\mathbf{k}^e = \int_v \mathbf{B}^T \left(\bar{\mathbf{E}} + \Delta\mathbf{E}\right)\mathbf{B}\, dv = \bar{\mathbf{k}}^e + \Delta\mathbf{k}^e \quad (15)$$

It should be noted that $\bar{\mathbf{E}}$ in Eq. (15) is different from $\mathbf{E}_{\det}$ in Eq. (14). The difference between these two matrices comes from the integration on the even-power of stochastic field functions, which is known as the stochastic integration (or weighted integral). The stochastic integration is defined as following:

$$X_{ij(k)} = \int_\Omega f_v^k p_i p_j d\Omega_e \qquad (16)$$

In Eq. (16), p_i, p_j are independent polynomials with which the strain-displacement matrix is decomposed into constant sub-matrices multiplied by a corresponding polynomial p_i, i.e., $\mathbf{B} = \mathbf{B}_i p_i$, $i = 1, \cdots, n$ and n = number of independent polynomials in the strain-displacement matrix. The stochastic integral in Eq. (16) is taken as random variables. The random variable is constructed as 4 kinds since in the previous steps the expansion is truncated after 4th order.
In fact, only the deviator stiffness in the element stiffness in Eq. (15) contains the random variables leading the element stiffness to be a function of random variable themselves.

4 Evaluation of Random Response

4.1 Taylor's Expansion of Displacement

Since the element stiffness is a function of random variable X, the global stiffness and further the displacement vector are also functions of random variable X. Accordingly, the displacement vector $\mathbf{U}$ can be expanded with respect to the mean random variable $\bar{X}$ as

$$\begin{aligned} \mathbf{U} &\cong \mathbf{U}\big|_{\bar{\mathbf{X}}} + \delta X_r^e\, \mathbf{U},_{(er)}\big|_{\bar{\mathbf{X}}} \\ &= \mathbf{U}_{\bar{\mathbf{X}}} - \bar{\mathbf{K}}^{-1}\delta X_r^e\, \mathbf{K},_{(er)}\big|_{\bar{\mathbf{X}}}\, \mathbf{U}_{\bar{\mathbf{X}}} \end{aligned} \qquad (17)$$

If we note that the random variables in each finite element are different in its form depending on the power of stochastic field function, E.q (17) can be rewritten as the following equation:

$$\mathbf{U} \cong \mathbf{U}_{\bar{\mathbf{X}}} - \bar{\mathbf{K}}^{-1}\sum_{k=1}^{4}\left\{\delta X_r^e\, \mathbf{K},_{(er)}\big|_{\bar{\mathbf{X}}}\right\}_{(k)} \mathbf{U}_{\bar{\mathbf{X}}} \quad (18)$$

where $\bullet|_{\bar{\mathbf{X}}}$ signifies the evaluation at the mean random variable vector $\bar{\mathbf{X}}$. $[],_{(er)}$ denotes derivative of a quantity with respect to the r-th random variable that belongs to element 'e' , X_r^e .The summation index k is used because the kinds of random variables is 4. $\bar{\mathbf{K}}$ is the mean stiffness matrix which includes the effect of the contribution from the integration on the even-power stochastic field function.

4.2 Mean and Covariance of Response

Since we have the displacement expanded with respect to the mean random variable, the mean and covariance of the displacement can be evaluated by applying the mean and covariance operator as follows:

$$E[\mathbf{U}] = \mathbf{U}_{\bar{\mathbf{X}}} \qquad (19)$$

$$Cov[\mathbf{U}, \mathbf{U}] = E\left[\delta\mathbf{U}\delta\mathbf{U}^T\right] \qquad (20)$$

In Eq. (20), the deviator displacement vector is obtained to be $\delta\mathbf{U} = \mathbf{U} - \mathbf{U}_{\bar{\mathbf{X}}}$. From Eq. (18) we can note that the deviator displacement consists of three different components depending on the kinds of random variables. Using brief notation, the deviator displacement is

$$\delta\mathbf{U} = -\mathbf{V}_A + \mathbf{V}_{(ii)} + \mathbf{V}_{(iv)} \qquad (21)$$

where the subscripts $A, (ii), (iv)$ means all random variables, random variables with f_v^2 and f_v^4, respectively. Accordingly, the covariance is given as follows:

$$\begin{aligned} Cov[\mathbf{U}] &= \varepsilon\left[\mathbf{V}_A\mathbf{V}_A^T\right] - \varepsilon\left[\mathbf{V}_{(ii)}\mathbf{V}_{(ii)}^T\right] \\ &\quad - \varepsilon\left[\mathbf{V}_{(iv)}\mathbf{V}_{(iv)}^T\right] - 2\varepsilon\left[\mathbf{V}_{(ii)}\mathbf{V}_{(iv)}^T\right] \end{aligned} \qquad (21)$$

5 Example Analyses

5.1 Expression for Stress Resultants

The example structure has dimensions of 10×10, and the material constants are $E_1 =$ 125Gpa, $E_2 =$ 9.87GPa, the Poisson's ratio $\overline{\nu}$ = 0.32, $G_{23} = G_{31}$ = 5.92GPa, G_{12} =6.58GPa. In order to investigate the in-plane behavior, only the in-plane distributed load of 100 per unit length in y-direction is applied. The thickness of plate is assumed to be 0.2. As boundary conditions, only x- and y-displacements are assumed as released in the corresponding sides of the plate. The lamination schemes are denoted by (Θ/- Θ)c, where Θ denotes stacking angle and 'c' is 'a' (asymmetric stacking) or 's' (symmetric stacking). The coefficient of variation (COV) of the stochastic field is assumed to be 0.1.

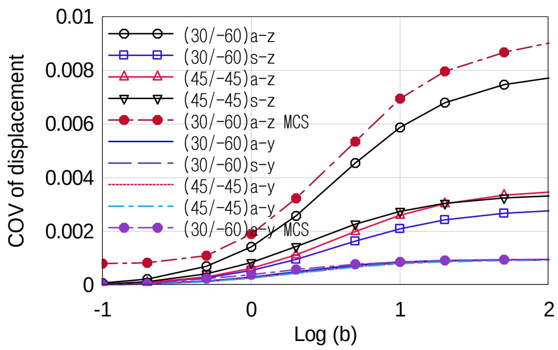

Fig. 1. COV in z- and y-displacement

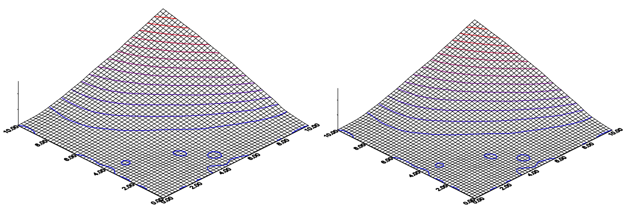

Fig. 2. 3D plot of standard deviation (SD) for MCS (left) and proposed scheme (right)

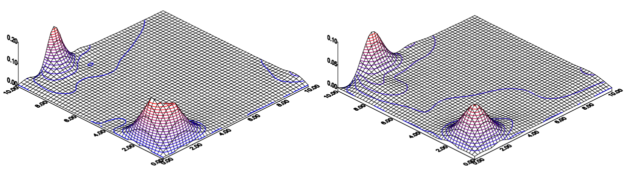

Fig. 3. 3D plot of coefficient of variation (COV) for MCS (left) and proposed scheme (right)

Fig. 1 shows the COV variation of the displacement, which is obtained to be about 1% of the stochastic field COV in the loading direction (y) and about 8% in out-of-plane direction (z) at the right-upper corner. In case of in-plane loading direction displacement, the COVs are not affected by the lamination scheme. However, in case of out-of-plane direction the asymmetric lamination is revealed to be relatively pronounced if compared with the other cases. The degree of agreement is better for in-plane loading direction displacement than the out-of-plane component.

The distribution of the standard deviation (SD) in z-displacement over the domain of the example plate is shown in Fig. 2. As seen in Fig. 2, the proposed scheme shows good agreement with the MCS.

Conclusions

The uncertain response of composite plate, especially in the in-plane behavior, due to random Poisson's ratio is given. The proposed scheme is revealed to be in good agreement with MCS.

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