HOMOGENIZATION AND DE-HOMOGENIZATION OF FIBER REINFORCED COMPOSITE LAMINA

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1 Abstract

The goal of the current work is to determine the validity of the micromechanical enhancement method in composite laminates containing large gradients of strain (macro-gradients), such as those developed at the free-edge of dissimilar lamina [6]. It has been shown that analyzing lamina as Cauchy continua can lead to erroneous results in the presence of macro-gradients and that micro-polar elasticity is a more appropriate model in these regions [1, 3, 4, 5]. Although the micromechanical enhancement method is a computationally efficient method for recovering micro-level information, it is limited by the same assumption of uniform macro-fields underlying homogenization via Cauchy elasticity. This leads to the question: is micromechanical enhancement capable of recovering micro-fields in regions of large macro-gradients or is a more complete representation of the deformation required? To answer this, results are compared between a fully heterogeneous solution; a micromechanically enhanced solution that neglects bending effects and a micromechanically enhanced solution that accounts for bending effects. The inclusion of bending effects in the micromechanical enhancement procedure is new to this work and is achieved through the application of additional deformation states to the heterogeneous unit cell.

By analyzing the Pagano-Rybicki problem, we find that macro-gradients do not have a significant influence on the maximum value of the dilatational, J_1, or the distortional, e_{vM}, strain invariants calculated in the critical unit cell but do significantly influence the distribution of these fields. It is noted that this result is problem specific and a broader class of problems should be analyzed before this conclusion is drawn in general. It is found that if one desires an accurate distribution of stresses and strains within the matrix phase, then a sub-modeling technique should be employed.

2 Introduction

The current work is focused on the multi-scaled analysis of fiber reinforced polymer matrix composite materials. The defining features of these materials, which complicates an analysis, is the large variability in microstructure and scales. Depending on the specific materials used, there can be one (boron filament) to several dozen (carbon fiber) reinforcements contained through the thickness of the lamina. These laminae are then oriented and stacked together to form a laminate. A laminate can be composed of any number of lamina, the upper limit being controlled by the manufacturing process [2]. Therefore, the ability to model every fiber in real world structures is often unfeasible due to the large disparity in scales - fiber diameters with dimensions in micrometers to aircraft wings with dimensions in tens of meters. To connect these scales, we often turn to micromechanics.

Within the field of micromechanics there are two major problems, homogenization and de-homogenization. The term "homogenization" is applied to the process of determining the effective properties of the equivalent homogeneous medium while the term "de-homogenization" is used to describe methods of recovering the relevant fields, e.g. stress or strain, at the constituent level based on the response of the equivalent homogeneous medium.

In this work we analyze the well-known Pagano-Rybicki problem using the micromechanical enhancement method [4, 5, 6]. Specifically, we will study in detail the problem posed by Hutapea et. al, of a laminate with four rows of fiber reinforcement and two unreinforced lamina. The fibers are arranged in a uniform square grid, have a diameter of 100 μm and are spaced to obtain a fiber volume fraction of 30.7%. A uniform strain of 0.2% is applied in the fiber longitudinal direction resulting in a state of generalized plane strain. Results will be compared in the critical unit cell, see Figure 1 [4].

For comparison, two analyses will be performed. The first analysis models the heterogeneous domain with the fiber and matrix phases described explicitly. This analysis will be referred to as the micromechanics (MM) solution. The second analysis uses the classical homogenization method, termed effective modulus (EM), at the lamina level. These results are de-homogenized with the micromechanical enhancement method described in [6] and compared to the MM solution.

In this second analysis, the effective moduli are obtained by solving six boundary value problems on a heterogeneous unit cell. Thermal effects are neglected in the current work but can be accounted for through the solution to an additional boundary value problem [6]. The laminate is analyzed as a bi-material plate composed of homogeneous lamina. Strains and stresses are recovered from the EM solution by calculating the volume average of the six strain components in the critical cell. We then take a linear combination of the states of strain in the unit cell obtained during the homogenization analysis. In this way, the de-homogenization step is reduced to a matrix multiplication and additional elasticity solutions do not need to be obtained. In a sub-modeling approach, one would calculate homogenized lamina properties, perform the EM analysis and then de-homogenized by applying either forces or displacements as boundary conditions to a unit cell. This approach requires solving a new boundary value problem for the de-homogenization step.

The work of Hutapea et al., concluded that the assumption of uniform macro-fields used in the homogenization method described above leads to erroneous results for problems containing large macro-stress gradients [4]. To relieve this problem, the previous work used micro-polar elasticity in the homogenization step and a displacement based sub-modeling technique to accurately recover stresses at the fiber/matrix interface in the critical unit cell. The use of micro-polar elasticity was shown to greatly increase the accuracy of the predicted state of stress at the fiber/matrix interface [4].

In this work, we would like to determine if the inclusion of volume average strain gradients in the micromechanical enhancement method provides a more accurate estimate of the state of strain in the critical unit cell. This is achieved by the addition of two bending deformation modes in the de-homogenization step only. Results within the critical unit cell will be compared for three different sets of micromechanically enhanced solutions. The first (MME1) will follow the uniform deformation assumption described in [6] in which bending is neglected and volume averages are calculated from the EM solution. The second (MME2) will include volume average bending components calculated from the EM solution. The final MME results (MME3), are obtained by calculating the volume average strains and curvatures from the MM solution. The MME3 solution will show that the least accurate approximation in the micromechanical enhancement method is independent of the homogenization step. Comparison of results obtained directly from the MM solution and the three de-homogenized results will indicate which assumptions lead to the greatest reduction of accuracy in the micromechanical enhancement method.

3 Micromechanical Enhancement

3.1 Homogenization

As stated earlier, the purpose of the homogenization procedure is to obtain effective material moduli. These moduli relate the volume average stress components to the volume average strain components. Using the contracted index notation, the desired relationship is given in Equation 1.

$$\bar{\sigma}_i = \bar{C}_{ij}\bar{\varepsilon}_j \quad (i, j = 1-6) \tag{1}$$

The over-bar indicates volume average stress and strain components and homogenized stiffness terms. In the method presented by Ritchey et. al, six independent states of volume average strain are applied to the unit cell through the application of mixed boundary conditions. The resulting volume average stress components are calculated from the reaction forces for each of the six problems. Using the superscript to indicate the loading case, we can write

$$\bar{\sigma}_i^{(k)} = \bar{C}_{ij}\bar{\varepsilon}_j^{(k)} \; (i,j,k=1-6) \tag{2}$$

If the columns of volume average strain matrix, $\bar{\varepsilon}_j^{(k)}$, are linearly independent, i.e. we have applied 6 independent states of strain, then $\bar{\varepsilon}_j^{(k)}$ is invertible and both sides of Equation 2 can be right multiplied by this inverse. Ritchey et. al, gave a set of displacement boundary conditions such that the $\bar{\varepsilon}_j^{(k)}$ matrix is a multiple, ε, of the identity matrix. Under this condition, each column in the effective stiffness matrix is determined by:

$$\bar{C}_{ik} = \bar{\sigma}_i^{(k)}/\varepsilon \; (i,k=1-6) \tag{3}$$

A typical choice of ε is one unit of micro-strain, i.e. $\varepsilon = 10^{-6}$.

3.2 De-homogenization

First, consider an unit cell subjected to a uniform extension in the longitudinal fiber direction such that we obtain $\bar{\varepsilon}_1 \neq 0$ in the absence of the other volume average strain components. At any point within the unit cell we will have a general state of strain given by:

$$\varepsilon_i^{(1)}(x_1,x_2,x_3) = m_i^{(1)}(x_1,x_2,x_3)\bar{\varepsilon}_1 \tag{4}$$

Clearly, the terms in the coefficient vector, $m_i^{(k)}$, are determined by normalizing the strain at any point in the unit cell, ε_i, by the applied volume average strain, $\bar{\varepsilon}_1$. In a similar manner, we can apply a uniform extension transverse to the fiber direction to obtain $\bar{\varepsilon}_2 \neq 0$ in the absence of the other volume average strain components. Under this new loading condition we have:

$$\varepsilon_i^{(2)} = m_i^{(2)}\bar{\varepsilon}_2 \tag{5}$$

For brevity, terms without over bars are implied to be functions of the unit cell coordinates. The principal of linear superposition can be stated as: the state of strain in a domain due to a combined loading state is equal to the sum of the strains due to the individual loading states. Therefore, if we wish to determine the state of strain at any point within the unit cell due to a linear combination of $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$, we need only add the contributions due to each of the individual loading conditions.

$$\begin{aligned} \varepsilon_i^{(1+2)} &= \varepsilon_i^{(1)} + \varepsilon_i^{(2)} \\ \varepsilon_i^{(1+2)} &= m_i^{(1)}\bar{\varepsilon}_1 + m_i^{(2)}\bar{\varepsilon}_2 \end{aligned} \tag{6}$$

By assuming that the state of strain at any point within the unit cell results exclusively from the application of volume average strain components, we may write:

$$\varepsilon_i = M_{ij}\bar{\varepsilon}_j \; (i,j=1-6) \tag{7}$$

Where the columns of the M_{ij} matrix are composed of the $m_i^{(k)}$ vectors. Following the homogenization method, for six independent states of volume average strain we have:

$$\varepsilon_i^{(k)} = M_{ij}\bar{\varepsilon}_j^{(k)} \; (i,j,k=1-6) \tag{8}$$

By using the same six boundary value problems that were proposed for the homogenization method, the columns in the M_{ij} matrix are determined by:

$$M_{ik} = \varepsilon_i^{(k)}/\varepsilon \; (i,k=1-6) \tag{9}$$

In addition to the uniform extension and shear modes, we can also include bending modes. To this end, we need to determine the volume average curvature within a unit cell. For generalized plane strain problems, the non-zero curvatures are:

$$\begin{aligned} \kappa_{31} &= \phi_{1,3} = \frac{\partial}{\partial x_3}\left[\frac{1}{2}\left(u_{3,2}-u_{2,3}\right)\right] \\ \kappa_{21} &= \phi_{1,2} = \frac{\partial}{\partial x_2}\left[\frac{1}{2}\left(u_{3,2}-u_{2,3}\right)\right] \end{aligned} \tag{10}$$

The volume average of a curvature is determined by applying the Gauss theorem twice, for example:

$$\begin{aligned} \bar{\kappa}_{31} &= \frac{1}{V}\int_V \kappa_{31}\,dV \\ &= \frac{1}{2V}\int_{-L_1/2}^{+L_1/2}\Big(u_3\big|_{(+L_2/2,-L_3/2)} + u_3\big|_{(-L_2/2,+L_3/2)} \\ &\qquad - u_3\big|_{(-L_2/2,-L_3/2)} - u_3\big|_{(+L_2/2,+L_3/2)}\Big)dx_1 \end{aligned} \tag{11}$$

Where the subscripts indicate the face on which the function is evaluated, see Figure 2. A volume average κ_{31} can be obtained by applying a linear gradient in the u_3 displacement as:

$$u_3(x_1,x_2,\pm L_3/2) = \pm\kappa x_2 \tag{12}$$

The other displacement boundary conditions applied to the unit cell are:

$$
\begin{aligned}
&u_1\left(\pm L_1/2, x_2, x_3\right)=0 \\
&u_2\left(x_1, 0, 0\right)=0
\end{aligned} \tag{13}
$$

All other degrees of freedom are traction free. These boundary conditions represent a pure bending mode of deformation on the single unit cell. Additional layers of unit cells could be included in the bending analysis. In this case, the central cell will be under pure bending while the additional cells will be subjected to both bending and a uniform extension.

Once the strains have been recovered at the micro-level they need to be compared to the MM solution to determine the nature of the approximations made. For this purpose, we will compare the dilatational, J_1, and the distortional, e_{vM}, strain invariants. These measures are selected as they provide a measure of both volume change and shape change at a point within the critical unit cell. The invariants are defined by:

$$
\begin{aligned}
&J_1=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} \\
&3J_2^{'}=\frac{1}{3}\left[\left(\varepsilon_{xx}-\varepsilon_{yy}\right)^2+\left(\varepsilon_{yy}-\varepsilon_{zz}\right)^2+\left(\varepsilon_{zz}-\varepsilon_{xx}\right)^2\right] \\
&\qquad +\frac{3}{4}\left[\gamma_{yz}^2+\gamma_{xz}^2+\gamma_{xy}^2\right] \\
&e_{vM}=\sqrt{3J_2^{'}}
\end{aligned} \tag{14}
$$

The assumptions implicit in the micromechanical enhancement method are: the volume average strain obtained from an effective modulus solution is equal to the volume average strain in the heterogeneous domain; the state of strain at any point within the unit cell is determined by the volume average strain components; the constituent materials are linear elastic and no large scale deformations occur.

4 Results

Following the work of Hutapea et. al, we are using isotropic silicon carbide fibers (E = 325 GPa, ν = 0.15) embedded in an isotropic epoxy matrix (E = 3.45 GPa, ν = 0.35) with a fiber volume fraction of 30.7%.

4.1 Homogenization

Due to the uniform arrangement of the fibers in the lamina, a cubic unit cell is selected. The homogenization method described in Section 2.1 is used to determine the terms in the effective stiffness matrix for this unit cell. Table 1 gives a comparison of the effective moduli given by Hutapea et. al, and those calculated using the current method. This comparison shows that both methods give equivalent results.

4.2 Micro-Fields

Table 2 shows the volume average strain and curvature components in the critical cell calculated from the MM and the EM solutions. This comparison shows that the EM solution provides a reasonable estimate of the $\bar{\varepsilon}_{xx}, \bar{\varepsilon}_{yy}, \bar{\varepsilon}_{zz}$ and $\bar{\kappa}_{zx}$ volume average components but does not predict the $\bar{\gamma}_{yz}$ and $\bar{\kappa}_{yx}$ volume average components. This result brings into question the validity of the EM solution.

Table 3 compares the limiting values (maximum and minimum) for the J_1 and the e_{vM} strain invariants obtained within the matrix phase of the critical unit cell. The MM results are used for comparison to three methods of de-homogenization. MME1 uses the EM volume average components without bending; MME2 uses the EM volume average components including bending and MME3 uses the volume average strains and curvatures obtained from the MM analysis. The MME3 solution contains the fewest assumptions of the micromechanical enhancement methods investigated. This comparison shows that all three de-homogenization methods provide reasonable estimates of the limiting values of the dilatational, J_1, and the distortional, e_{vM}, strain invariants.

These results also show that, for the current problem, the addition of the volume average curvatures has very little effect on the range of values obtained within the critical cell. Further, using the volume average strain and curvature components obtained from the MM solution does not improve the estimates. This indicates that the assumption that leads to the greatest overall reduction in accuracy is the use of volume average deformation modes in the de-homogenization step. Since this is the assumption that provides the computational efficiency to the method, it appears that the inclusion of an improved homogeneous solution, e.g. micro-polar elasticity, will not improve the final estimates.

Figures 3 and 4 compare the distributions of the J_1 and the e_{vM} strain invariants, respectively, obtained from the MM, MME1, MME2 and MME3 solutions. As expected, these plots show that the distribution of strain within the critical cell is highly dependent on the boundary conditions applied to the cell. Namely, the de-homogenized solutions do not account for the free-edge boundary condition on the positive y-face of the critical unit cell. Away from the free-edge, the distributions are similar for all four analyses.

5 Conclusions

The results of the current study show that, for the Pagano-Rybicki problem, the MME method provides a reasonable estimate of the maximum and minimum J_1 and e_{vM} strain invariants within the critical unit cell. These results also show that the error incurred in reducing all possible deformation states on a unit cell to volume average strains and curvatures outweigh the errors arising from the effective modulus theory. That is, although Hutapea et al., showed that micro-polar elasticity provides improved results when used with a sub-modeling technique, a similar improvement would not be obtained if used in conjunction with the micromechanical enhancement method. This result may not be general as the curvatures achieved in the critical unit cell are several orders of magnitude smaller than the maximum volume average strain component. Micro-polar effects are likely to be more relevant in problems with large macro-curvatures such as non-symmetric laminates and plate bending problems.

Several important conclusions can be drawn from the current work. First, if one desires to recover an accurate distribution of stress and strain in the fiber and matrix phases, then a sub-modeling technique is suggested. Since the goal of such an approach is to recover the fields exactly, accurate boundary conditions need to be applied to the sub-domain of interest. In regions of high macro-stress gradients, these boundary conditions can come from an EM solution if the sub-domain boundary is sufficiently far from the location in which high accuracy is desired [7]. The work of Hutapea et al., showed that smaller sub-domains may be selected if micro-polar elasticity is used at the lamina level [4]. This conclusion has only been validated for the relatively simple Pagano-Rybicki problem and further studies are required for this to be taken as a general principal.

The final conclusion which we will draw is that the micromechanical enhancement method provides reasonable bounds on the maximum and minimum values of the dilatational and distortional strain invariants for the problem investigated. Since the location of the limiting values is not predicted, this serendipitous result needs further validation in a larger class of problems. Despite this, the micromechanical enhancement method can lead a designer to areas of critical importance using micro-level information. With this information, a minimum set of high fidelity models can be analyzed.

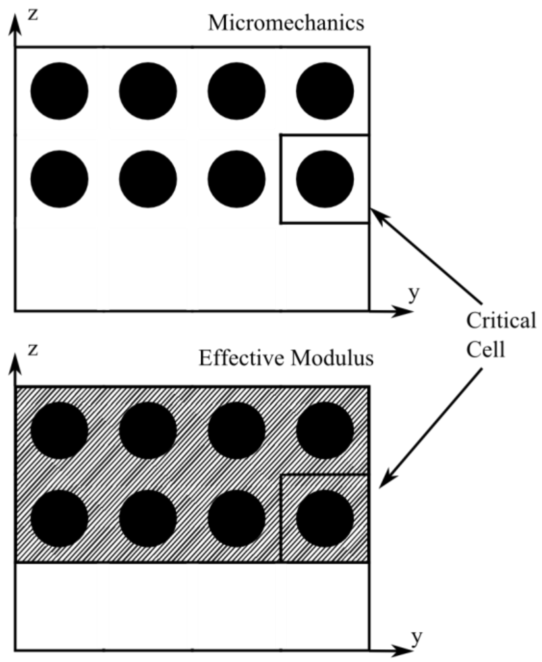

Fig.1. Schematic of the micromechanics (MM) and the effective modulus (EM) domains.

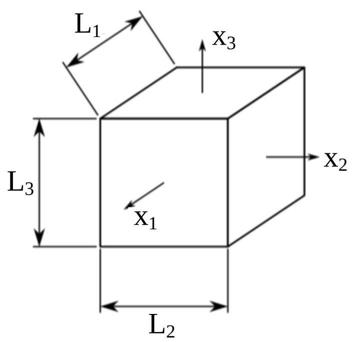

Fig.2. Unit cell coordinates and dimensions.

Property	Hutapea [4]	Current
E_1 (GPa)	102.11	102.15
$E_2 = E_3$ (GPa)	7.27	7.27
ν_{23}	0.440	0.440
$\nu_{13} = \nu_{12}$	0.277	0.277
G_{23} (GPa)	1.99	1.99
$G_{13} = G_{12}$ (GPa)	2.39	2.37

Tab.1. Effective moduli published by Hutapea et. al, compared those obtained in the current work

Component	MM	EM
$\bar{\varepsilon}_{xx}$ (10^{-6})	2000.0	2000.0
$\bar{\varepsilon}_{yy}$ (10^{-6})	-555.1	-565.1
$\bar{\varepsilon}_{zz}$ (10^{-6})	-556.2	-555.1
$\bar{\gamma}_{yz}$ (10^{-6})	2.4	18.5
$\bar{\kappa}_{zx}$ (10^{-6} m^{-1})	0.1344	0.1533
$\bar{\kappa}_{yx}$ (10^{-6} m^{-1})	-0.1102	-0.2918

Tab.2. Volume average of strain and curvature terms calculated in the critical cell of the heterogeneous and homogeneous solutions respectively.

Invariant	MM	MME1	MME2	MME3
J_1^{max} (10^{-3})	0.742	0.688	0.700	0.695
J_1^{min} (10^{-3})	0.591	0.616	0.615	0.636
e_{vM}^{max} (10^{-3})	2.782	2.772	2.773	2.758
e_{vM}^{min} (10^{-3})	2.661	2.657	2.651	2.654

Tab.3. Bounds of the J_1 and the e_{vM} strain invariants calculated in the matrix phase.

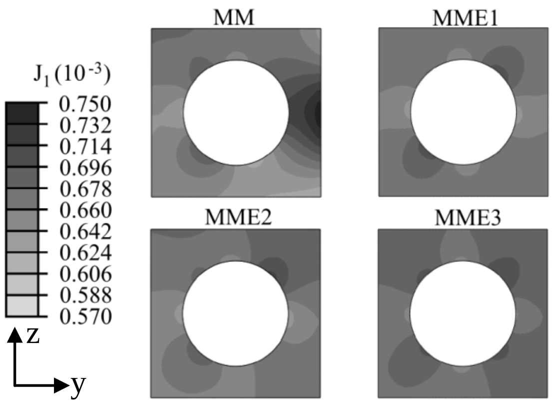

Fig.3. Distribution of the J_1 strain invariant in the critical unit cell

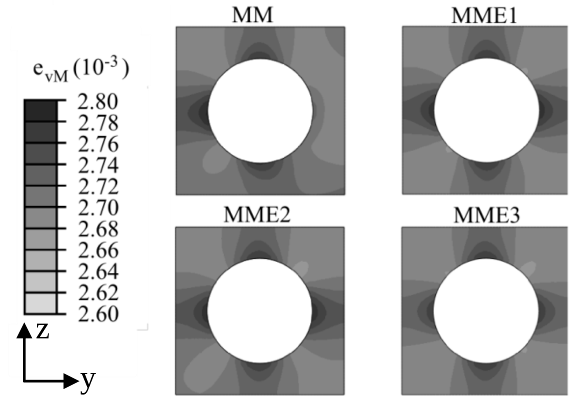

Fig.4. Distribution of the e_{vM} strain invariant in the critical unit cell

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