PREDICTING FATIGUE DAMAGE DEVELOPMENT FOR BRAIDED CARBON FIBER POLYMER MATRIX COMPOSITES

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1 Introduction

Continuous fiber reinforced polymer matrix composite (PMC) laminates have become indispensible for commuter aircraft structural applications, and are extensively used for manufacturing primary load-bearing structural components. The principal advantage of employing PMC laminates is the reduction in the weight of the vehicle, which leads to a superior fuel economy. However, a number of manufacturing and performance disadvantages have been identified with these 'conventional' laminates. This has subsequently prompted the development of fabric reinforced PMC materials including two-dimensional and three-dimensional braided composites. Some of the performance advantages identified with braided fabric reinforced composites include better overall through-the-thickness strength properties such as superior impact damage resistance and delamination resistance, balanced in-plane performance, improved fatigue performance and lower notch sensitivity [1]. In addition, producing complex shaped parts can be easier to manufacture and generally have a significantly lower manufacturing cost due to the conformability of braided fabrics coupled with the use of an out-of-autoclave manufacturing technique such as resin transfer moulding (RTM). The automation of the braided fabrics may further reduce manufacturing costs, eliminating tedious manual lay-up which renders these composites as cost-competitive alternatives for component manufacturing.

In spite of the indicated advantages, the use of braided PMC materials is only limited to a few applications in the aerospace industry to date. One of the issues restricting their wider use is that there have been few studies conducted to characterize or model their material behaviour [2],[3]. In addition, few of these studies attempt to consider the fatigue behaviour of braided PMCs. Before braided composites can be utilized by designers in the industry to assess the structural integrity and damage tolerance capabilities of components, viable prediction tools must be developed for these materials. With respect to PMC materials, predicting the microscopic damage development caused by cyclic loading is of paramount importance for developing damage tolerant structural components, and for predicting their fatigue life.

There have been many different types of PMC fatigue prediction models reported in the literature [4]-[9], which illustrates that one common fatigue prediction tool has not yet been accepted or established in the industry. Currently, composite components are designed too conservatively due to inadequate prediction tools and the deficiency of expertise available in the industry. There is a need to develop simple and viable fatigue prediction tools for PMC components. It is therefore the objective of this study to develop a fatigue prediction model for a triaxially braided carbon fiber reinforced PMC material. The subsequent sections outline the details of the prediction model development and finite element implementation, and present the prediction model results as well as the conclusions.

2 Model Details

Many fatigue prediction models developed for PMC materials are based on the fundamental concepts of damage mechanics [10]. These prediction models consider the effects of particular microscopic damage mechanisms such as matrix cracking and debonding, and represent them as a set of internal state variables by scalars, vectors or tensors within the constitutive equations [11]-[13]. Damage is assumed to be homogenized or diffused within the bulk material, and the models are built within the framework of continuum damage mechanics (CDM). The formulation of the 'damaged' constitutive equations is based on the concepts of irreversible thermodynamics in that damage events are irreversible microscopic processes within the bulk material. The damage parameters are typically incorporated within the material stiffness tensor, and thus the degradation of the mechanical properties is directly modeled. The damage parameters are often derived from experimental data, but can also be defined by analytical expressions or from micromechanical analyses. The CDM-based fatigue prediction model developed in this study incorporates scalar damage variables in the constitutive equations, and a cumulative damage law that is a function of the number of fatigue cycles to define the evolution of the damage variables and ultimately the degradation of the effective material properties. The prediction model is partly based on the work by Ladeveze and LeDantec [11], and utilizes the concept of effective stress and the principle of strain equivalence.

2.1 Damage Model Formulation

The undamaged triaxially braided PMC material is considered to behave as an orthotropic linearly elastic material, and the diffuse damage is assumed to be evolving incrementally. Damage is also assumed to be orthotropic. Since in-plane tensile loading is considered, the braided composite is subjected to a plane stress state. According to CDM theory, three internal variables representing damage are incorporated to degrade the material elastic constants for the plane stress condition. The stiffness degradation expressions are given as:

$$\begin{aligned} E_{11} &= E_{11}^{o}\left(1-D_{11}\right) \\ E_{22} &= E_{22}^{o}\left(1-D_{22}\right) \\ G_{12} &= G_{12}^{o}\left(1-D_{12}\right) \end{aligned} \tag{1}$$

where E_{11} and E_{22} are the Young's modulii, G_{12} is the in-plane shear modulus, and D_{ij} are the corresponding internal damage variables. The superscript ' ^{o} ' refers to the undamaged material parameter.

The thermodynamic potential or free energy ($\rho\Psi$) is defined for a damaged orthotropic material in a plane stress state by the polynomial:

$$\begin{aligned} \rho\Psi = & \frac{1}{2}\left(\frac{E_{11}^{o}\left(1-D_{11}\right)}{1-\nu_{12}\nu_{21}}\right)\varepsilon_{11}^{2} + \\ & \frac{1}{2}\left(\frac{E_{22}^{o}\left(1-D_{22}\right)}{1-\nu_{12}\nu_{21}}\right)\varepsilon_{22}^{2} + \\ & \frac{1}{2}G_{12}^{o}\left(1-D_{12}\right)\gamma_{12}^{2} + \\ & \frac{\nu_{12}E_{22}^{o}\left(1-D_{22}\right)}{1-\nu_{12}\nu_{21}}\varepsilon_{11}\varepsilon_{22} \end{aligned} \tag{2}$$

where ν_{12} and ν_{21} are the major and minor Poisson ratios, respectively, ε_{11} and ε_{22} are components of the strain tensor, and γ_{12} is the engineering shear strain. It is assumed in this study that the damage terms D_{11} and D_{22} are only effective when the corresponding strains are positive, i.e., E_{11} and E_{22} are not affected during compressive loading states where the effects of damage are not influential on damage evolution [11].

From the thermodynamic potential of the damaged material, the resulting stiffness tensor is defined as:

$$C_{ij} = \begin{bmatrix} \frac{E_{11}^{o}\left(1-D_{11}\right)}{\left(1-\nu_{12}\nu_{21}\right)} & \frac{\nu_{12}E_{22}^{o}\left(1-D_{22}\right)}{\left(1-\nu_{12}\nu_{21}\right)} & 0 \\ & \frac{E_{22}^{o}\left(1-D_{22}\right)}{\left(1-\nu_{12}\nu_{21}\right)} & 0 \\ \text{sym} & & G_{12}^{o}\left(1-D_{12}\right) \end{bmatrix} \tag{3}$$

2.2 Fatigue Damage Evolution

The damage kinetics for cyclic loading (i.e., $\partial D_{ij}/\partial n$) can be defined by the corresponding thermodynamic forces (i.e., $\partial\Psi/\partial D_{ij}$) or directly from experimental data. In this study, the evolution of the D_{ij} terms is based on quantitative measurements of the underlying damage mechanisms as well as

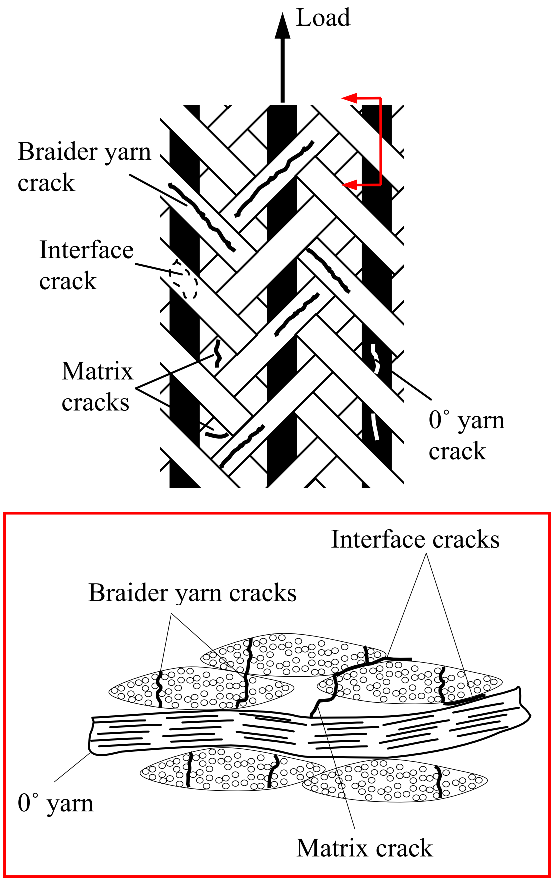

Fig. 1. Schematic of observed damage mechanisms.

qualitative observations obtained through the experimental characterization of test coupons for the triaxially braided PMC material [14]. In this study, crack density profiles for various damage mechanisms are defined at various applied cyclic stresses, including cracks within the braider yarns and cracks at the yarn interfaces as shown in the schematic in Fig. 1. The corresponding crack density profiles are used to define the evolution of the damage variables as a function of the number of loading cycles, n, providing the required damage kinetic, $\partial D_{ij}/\partial n$.

The expression for the evolution of the damage variables is accordingly defined as:

$$D_{ij} = \alpha_{ij}\left(\alpha_1 {\rho_{by}}^{\beta_1} + \alpha_2 {\rho_{\text{int}}}^{\beta_2}\right) \tag{4}$$

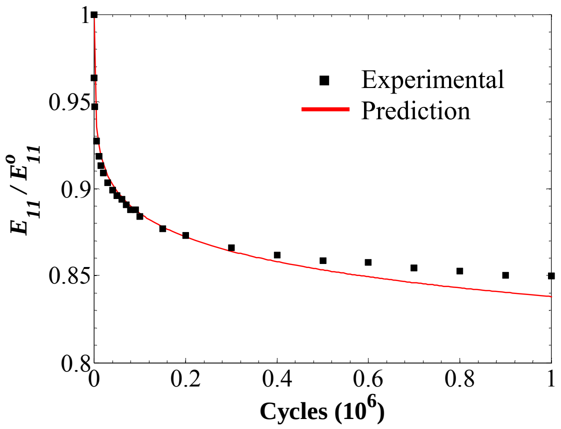

Fig. 2. Axial stiffness degradation.

where ρ_{by} and ρ_{int} are phenomenological crack density parameters that are functions of the number of loading cycles and the maximum applied stress, α_1, α_2, β_1, β_2 are material constants, and α_{ij} are scalar parameters that correspond to a particular material direction. The α_{ij} terms ensure that the effects of damage development correspond to the material symmetry, and thus the development of damage is also orthotropic.

In order to illustrate the accuracy of the damage kinetic defined by Eqn. (4), an analytical solution for a uniaxial stress state is used to compare with uniaxial tension-tension fatigue experimental test data [14]. The damage model is used to predict the degradation of the axial modulus (E_{11}) during constant amplitude load-controlled cyclic loading. A plot of the predicted and measured normalized axial stiffness (E_{11}/ E^{o}_{11}) as a function of the number of loading cycles for a maximum applied stress of 70% UTS is shown in Fig. 2. The damage model accurately captures the rapid stiffness degradation during the initial stage of cycling and the more gradual stiffness degradation thereafter, which was exhibited by the triaxially braided PMC material for all maximum applied stresses.

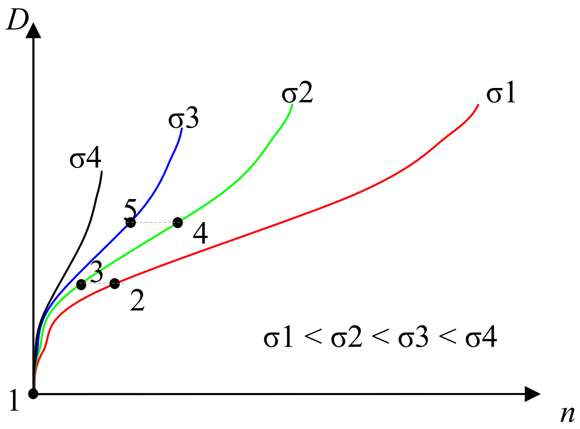

Fig. 3. Schematic of *D*-*n* curve interpolation.

2.3 Additional Model Considerations

When a PMC material is loaded in tension with a constant amplitude cyclic load, local damage development can lead to local stress redistributions. For a structural component, a variation in damage development due to the non-uniform stress field will cause local stress redistribution and a change in the local peak stresses even thought the maximum applied load is constant throughout cycling. The defined damage kinetic predicts the damage evolution as the number of loading cycles increases, but assumes a constant stress. Since the stress at a material point may change during cycling due to the aforementioned local stress redistribution, the material point may undergo variable amplitude cycling. This must be addressed by the prediction model.

One approach would be to utilize the set of damage kinetic curves defined by Eqn. (4), i.e., at different maximum applied stress levels, within an interpolation scheme to account for variable amplitude cyclic loading. Consider a schematic set of curves presented in Fig. 3 which represent damage development in a material direction for test specimens cycled with various constant amplitude maximum stress levels. As an example, if a material point for a structure is cycled with a maximum stress of σ1 for 50 cycles reaching a cumulative damage of 0.10 (i.e., point 1 to 2 in Fig. 3). Then, due to local stress redistribution, the element is subjected to a stress of say σ2 for 300 cycles. Now, the σ2 *D-n* curve can be invoked to determine the cumulative damage for the subsequent 300 cycles. The cumulative damage magnitude of 0.10 can be 'transferred' to the σ2 curve (i.e., point 3), and that curve can be utilized to define the cumulative damage for the remaining 300 cycles (until point 4). This process can be repeated as the local stress magnitude changes. Although this technique has never been employed for a fatigue damage simulation, observations of stiffness degradation in unidirectional-ply laminates subjected to variable amplitude cyclic loading supports the theory of this method [15].

Another important consideration for predicting the fatigue damage development of a structural component is the possibility of local multi-axial stress states. The damage kinetic obtained for the braided composite material (i.e., Eqn. (4)) is for uniaxial stress states. If a material point exhibits a multi-axial stress state, this must be accounted for in the prediction model. The damage kinetic must therefore be defined using a suitable expression that accounts for the corresponding multi-axial stress state. One approach is to define the load state by a set of scalar potential damage functions that are effectively the driving forces for damage in the different material directions. The damage functions can be defined in terms of stresses (or strains), and are formulated using the invariants of the corresponding stress (or strain) tensor.

One approach is to follow the strength-based failure criteria developed by Hashin [16], which utilizes the invariants of the stress tensor to define the failure criteria functions. These approaches are more physically based because they consider the material symmetry. Williams et al. [17] used a damage potential function which is analogous to a failure criterion function. The function is written in terms of local strain components and assumed to be an effective strain value. A similar function is adopted in this study and written in terms of local stress components, σ_{ij}. The damage function has the general form for an orthotropic material in a state of plane stress:

$$F = \sqrt{\left(\frac{\sigma_{11}}{K}\right)^2 - \left(\frac{\sigma_{11}}{K}\right)\left(\frac{\sigma_{22}}{L}\right) + \left(\frac{\sigma_{22}}{L}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2} \quad (5)$$

The constants K, L and S are scalars that are used to provide a measure of the relative contribution of each stress component to the driving force for damage growth in a particular material direction. The constants are defined from intuition based on the specific material. For example, consider the damage function in the material 11-direction (i.e., F^{11}). The scalar constant K^{11} is equal to 1 since σ_{11} is the primary driver of damage in the-11 material direction, while K^{22} should be less than 1 by this reasoning since σ_{11} is not the primary driver of damage in the 22-material direction. The damage function defined by Eqn. (5) is an effective stress which is used in the damage kinetic defined by Eqn. (4). Recall that the phenomenological crack density parameters are functions of the maximum applied stress.

3 Finite Element Implementation

The presented prediction model is implemented into the commercial finite element software package ANSYS through its user programmable features. ANSYS is a versatile software package that allows the user to define its own constitutive material behaviour through a user material (USERMAT) subroutine [18]. In this study, a FORTRAN-based USERMAT subroutine is developed in order to implement the material constitutive behaviour defined by Eqns. (1)-(5), which is linked within the ANSYS environment. The form of the constitutive model allows for a simple numerical implementation into the finite element software, and for direct use with displacement-based elements.

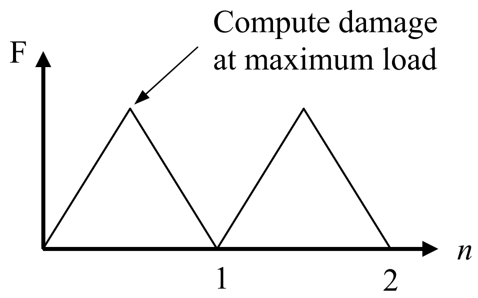

Fig. 4. Schematic of loading sequence for typical fatigue simulations.

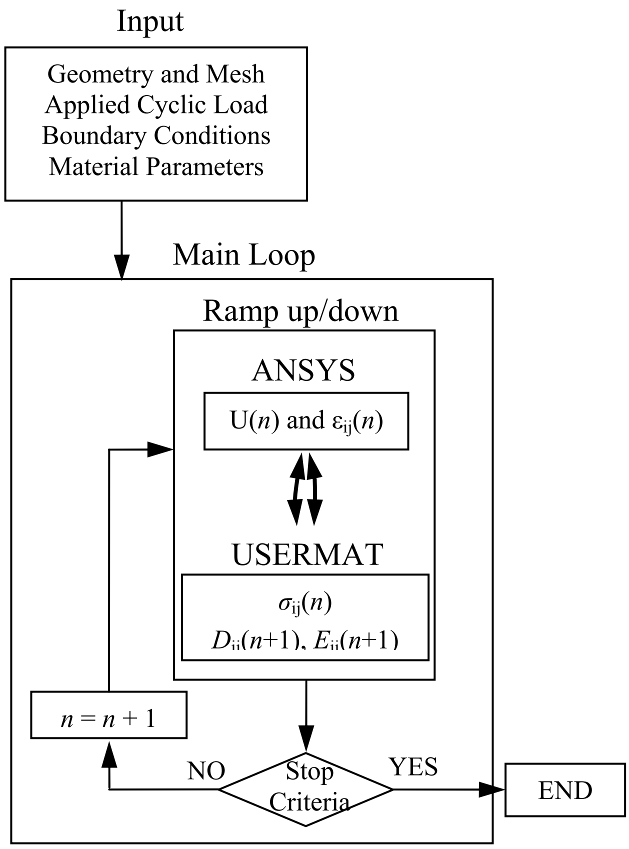

Fig. 5. Flow chart of the simulation scheme.

For numerical computations, the USERMAT subroutine is called at every material integration point in the element mesh during every solution load step, and possibly multiple times during one load step for the Newton-Raphson convergence iterations. In this study USERMAT is called for each material integration point twice during every simulated load cycle, once at the peak load and once at the minimum load. A schematic of the loading sequence for a typical simulation is shown in Fig. 4. At the peak load, the current damage state for each material point is computed and the degraded material constants are calculated for the next loading cycle. During the ramp-down to the minimum load, damage does not propagate. When called, the USERMAT subroutine requires stresses, strains and state variables (i.e., damage variables D_{ij}) at the beginning of the current load step for the corresponding material point, as well as the strain increments ($\Delta\varepsilon_{ij}$) determined by ANSYS as input. The subroutine is required to provide, as output, the values of the stresses at the end of the load step which are determined using the degraded material constants from the previous loading cycle and the current strain state. The subroutine also determines the values of the state variables and the updated stiffness tensor using Eqns. (4) and (3), respectively, which are used for the subsequent loading cycle (i.e., the next cycle ramp-up load step). This process is repeated for every loading cycle until a specified stopping criteria is met, which may be when a maximum loading cycle number or a limit value of the damage terms is reached. A flow chart of the simulation procedure is shown in Fig. 5. The subroutine allows storing of the damage variables and well as additional material parameters such as the stiffness tensor terms for post-processing.

4 Simulation Results

The goal of the study is to simulate fatigue damage development for a component manufactured from the triaxially braided carbon fiber reinforced PMC material [14]. In order to demonstrate the abilities of the proposed fatigue prediction model, a suitable flat component is studied and the results of the corresponding fatigue simulation are presented in this section. The geometry of the in-plane loaded flat plate containing a central hole is shown in Fig. 6. The plate width is 20 mm, the thickness is 3 mm and the hole diameter is 8 mm. Due to symmetry, only one quarter of the component is modeled, which is also shown in Fig. 6 along with the applied symmetry boundary conditions and the applied cyclic pressure load on the top edge of the component. A cyclic pressure load with a peak value of 55 MPa was used for the simulation, and the stress ratio was set to $R = F_{min}/F_{max} = 0.1$ to simulate tension-tension load controlled cyclic fatigue. The elements used for the simulation are 4-node quadrilateral elements with plane stress capabilities. The warp fiber direction (i.e., the material 11-direction) corresponds to the y-direction for the FE model which is the direction of the applied pressure load. The properties for the undamaged orthotropic material used for the analysis are scaled values of the true room temperature material properties, and are summarized in Table 1. The values of the damage coefficients presented in Eqn. (4) used in the simulation are summarized in Table 2. Note that the coefficients β_1 and β_2 are functions of the corresponding damage function defined by Eqn. (5), which is shown in Table 2.

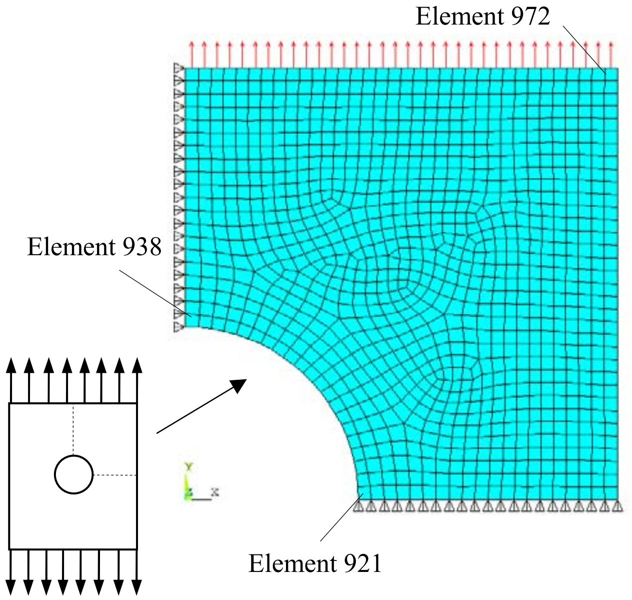

Fig. 6. Component geometry and FE mesh.

Table 1. Undamaged orthotropic material properties.

E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	ν_{12}
44	71	21	0.40

Table 2. Damage evolution coefficients.

α_1	α_2	β_1	β_2	α_{11}	α_{22}	α_{12}
0.001	0.001	2.4178 - 0.0021F	2.3125 + 0.0025F	1.00	0.70	0.89

(a)
(b)
(c)
(d)

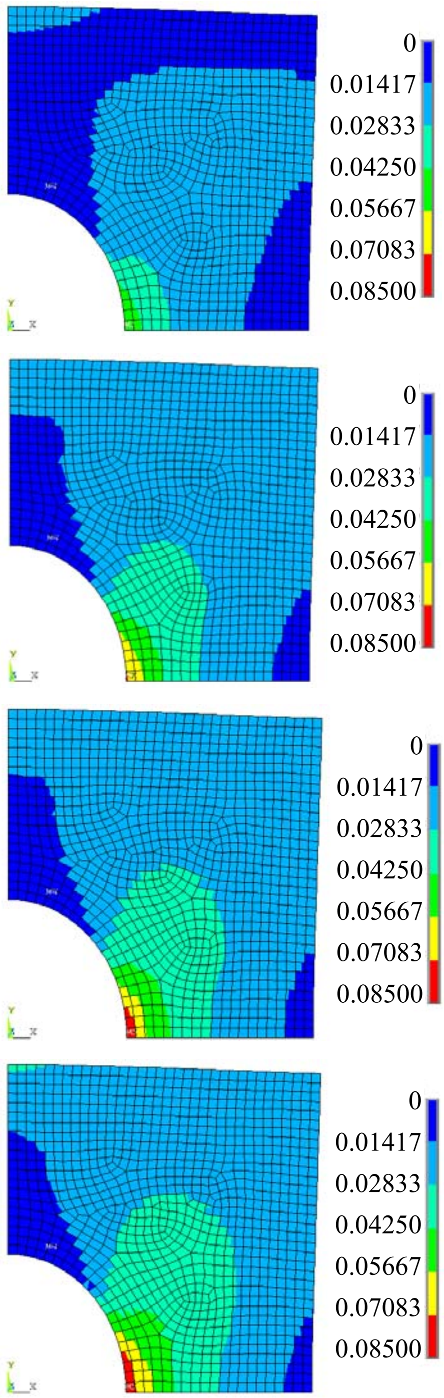

Fig. 7. Contour plots of D_{11} at loading cycle (a) 50, (b) 1000, (c) 2400, and (d) 5000.

In order to demonstrate the damage evolution prediction capabilities of the fatigue model, contour plots of the damage variable D_{11} are shown in Fig. 7

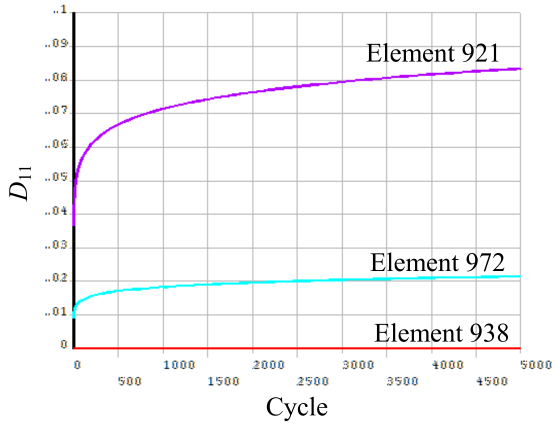

Fig. 8. D_{11}-n plots for indicated elements.

for the indicated simulation loading cycles. Recall that the material 11-direction is the direction of the applied pressure load. After only 50 loading cycles, the component has already exhibited a large degree of damage at the hole edge and in the surrounding regions. This of course is due to the stress concentrations at the hole. After 1000 loading cycles, damage is further developed in these same regions and has initiated in adjacent regions. This results from the stiffness degradation in the initially damaged regions, which consequently causes local stress redistributions and an increase in stress in the adjacent regions. The damage evolution in these adjacent regions consequently increases. Beyond 1000 loading cycles, damage increases at a more gradual rate which is evident after 2400 and 5000 loading cycles (see Figs. 7(a) and 7(b)).

A plot of the damage variable D_{11} as a function of the number of loading cycles is shown in Fig. 8 for three elements. These elements are identified within the FE mesh shown in Fig. 6. Element 921, which is located at the hole edge, exhibits a high degree of damage in the 11-material direction due to the high local stresses in the material 11-direction. On the other hand, element 972 exhibits significantly less damage due to much lower local stresses. The damage degradation rate after 1000 loading cycles is also much lower in element 972 compared to element 921. This is in line with the experimental fatigue results for the triaxially braided PMC material, which found that the damage degradation

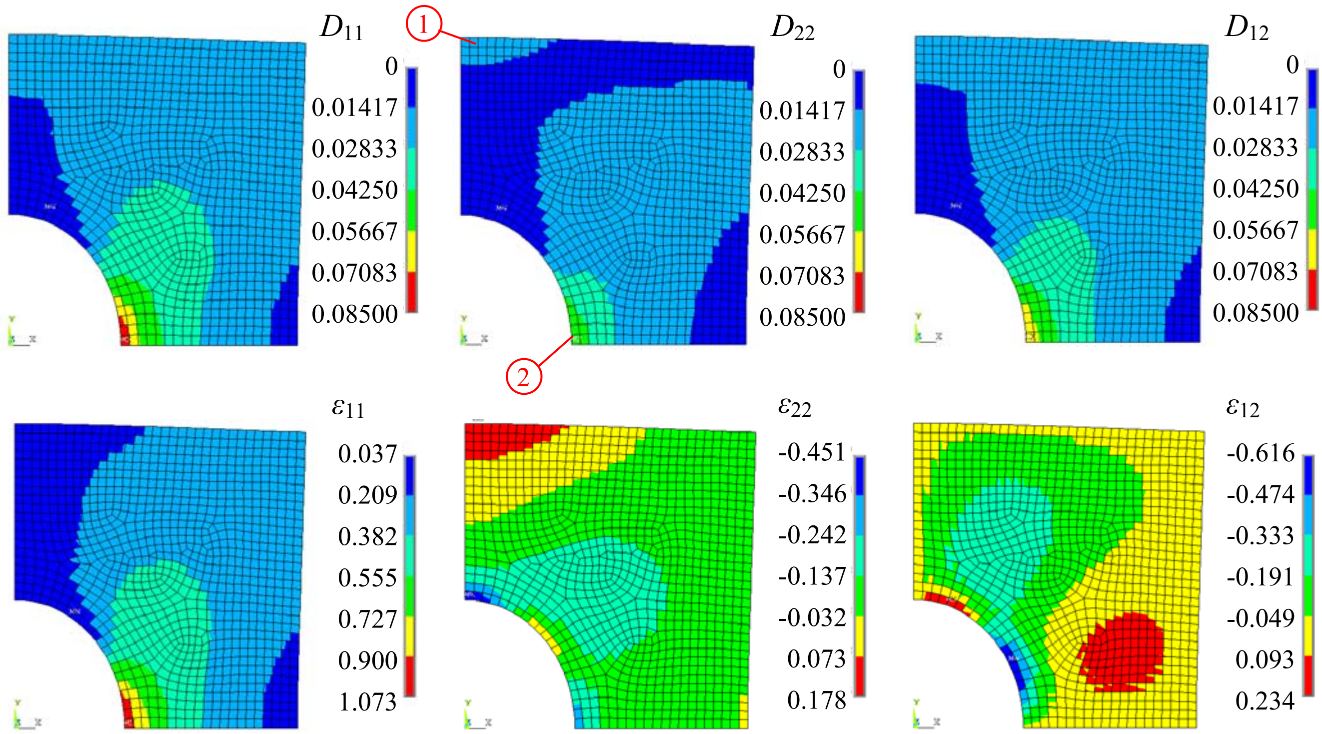

Fig. 9. Strain and damage components contours at 2400 cycles; strains are in units of %.

rate increased with increasing maximum applied stress [14]. Furthermore, the rapid increase in damage development during the initial stages of load cycling is seen in all elements subjected to positive stress levels, which also corresponds to the experimental results. Finally, element 938 exhibits negligible damage evolution due to the fact that the local stresses in that region of the component are compressive. Recall that the damage variables at a particular material point are not affected if that material point is subjected to compressive loading states.

Plots of the strain component contours and the damage variable contours are shown collectively in Fig. 9 at 2400 loading cycles. It is evident that the D_{11} damage component is evolving mainly due to the ε_{11} strain component and its influence on damage development in the material 11-direction. Note however that a positive ε_{22} strain component will also contribute to crack development along the braider yarns, which will contribute to development of the D_{11} damage component. This is illustrated by considering the ε_{11} and ε_{22} strain components in zone 1 of the component as indicated in Fig. 9. This region has positive ε_{22} strain which increases the D_{11} damage component in zone 1 as shown. The evolution of the D_{22} damage components in zone 2 are also mainly evolving due to the high ε_{11} strain components since the ε_{22} strain components are negative in this region. The D_{22} damage components in zone 1 are however evolving due to the high ε_{12} strain components in this region. The shear strains have less of an effect on the damage components, but are undoubtedly contributing to the damage evolution in zone 2 of the component. The contour plots in Fig. 9 illustrate the influence of the local strain components on the different damage variables, and show the ability of the prediction model in capturing this complex orthotropic damage development under local multi-axial stress states.

5 Conclusions

A fatigue model capable of predicting the complex orthotropic damage evolution under cyclic stresses was developed for a triaxially braided carbon fiber reinforced PMC material. The orthotropic material constitutive behaviour was developed within the framework of continuum damage mechanics, and the evolution of diffuse damage was introduced via scalar damage variables (with cyclic loading) that were defined using quantitative experimental damage formulations. The model was capable of accounting for local variable amplitude cyclic stress states due to load redistributions caused by evolving damage, as well as the influence of local multi-axial stress states on damage development. The fatigue simulations were conducted within the ANSYS environment using a developed user material subroutine which was linked to the finite element software. The results presented illustrated the predictive capabilities of the model. Although the model was used for two-dimensional displacement-based finite elements in a plane stress state, it may be extended for use with 3D displacement-based elements as well. The developed model is a viable tool for predicting fatigue damage development and the fatigue life of PMC components, and can easily be implemented into the finite element framework.

At this stage, the development of the prediction model is ongoing. Some future considerations include developing a scheme to easily define the damage evolution material coefficients. This may involve the use of a micromechanical finite element model of the braided composite in order to define the damage coefficients. Furthermore, the inclusion of compressive damage modes within the damage model may also be considered for components that are subjected to high compressive stresses. With regard to the fatigue simulation, 'cycle jumping' may also be considered in order to improve the efficiency of the simulations without affecting the convergence of the nonlinear solutions. Finally, in order to validate the predictive capabilities of the presented model, experimental tests on structural components manufactured from the braided composite material will be conducted. Digital image correlation may be used for these tests in order to yield surface strain maps of the components and a comparison with the simulation results.

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