THEORETICAL ANALYSIS ON CFRTP OPTIMAL STRUCTURE SUBJECTED TO BENDING LOAD AND THE INFLUENCE OF OUT-OF-PLANE SHEAR MODULUS

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ABSTRACT

In this study, optimum variable thickness of simple supported beam was theoretically obtained by solving the calculus of variations under the condition of same maximum deflection of flat beam. Under Euler-Bernoulli assumption, theoretical variable thickness was expressed by the function of two parameters, thickness of flat beam and length of the beam. In comparison with shape optimization by Finite Element Method (FEM), the calculation result was well consistent with theoretical solution except for the supporting edge, where FE meshing fineness affected. To examine the effect of the ratio of E_1 to G_{13}, Timoshenko assumption was also applied for theoretical analysis. Although explicit analytical solution was not obtained under Timoshenko assumption, numerical solution was obtained by inverse problem analysis. In comparison with the result under Euler-Bernoulli assumption, it was observed that the minimum thickness was required on the edge under Timoshenko assumption to meet with shear stress. Through the parametric study, it was observed that this minimum thickness on the edge was almost proportional to a square root of the ratio of E_1 to G_{13}. In comparison with flat beam having same rigidity, total weight reduction of variable thickness beam was approximately 16.0% under Euler-Bernoulli assumption and 15.0% under Timoshenko assumption. This difference was also caused by minimum thickness.

1 INTRODUCTION

Carbon fiber reinforced thermoplastics (CFRTP), the composite of carbon fiber and thermoplastic resin, are highly expected to apply for mass-production vehicle due to its high-productivity, thermal bonding characteristic and recycle performance. Although there are lots of precedence research on structural optimization of continuous carbon fiber reinforced thermosets (CFRTS) [1-4], in a practical sense such optimization for automobile parts has been limited to small parts such as stiffener position and hole shape, mainly owing to design constraint and aerodynamic characteristics. On the other hand, discontinuous CFRTP allows 'variable thickness structure' which enables wide range of practical application including large structural parts such as car floor.

One of the mechanical characteristics of CFRTP is the ratio of Young's modulus (E_1) to out-of-plane shear modulus (G_{13}). Due to the low shear modulus of thermoplastic resin, the ratio of E_1 to G_{13} of CFRTP is likely to be larger than other materials. In prior study, it has been reported the ratio of E_1 to G_{13} of CFRTP varies greatly depending on fiber morphology and heating condition [5-6], and this parameter affects the difference between actual and observed value of bending stiffness at the bending test [7]. Therefore, it is important to understand the influence of the ratio of E_1 to G_{13} for structural optimization of CFRTP. In this study, optimum variable thickness of simple supported beam was theoretically obtained by solving the calculus of variations under Euler-Bernoulli and Timoshenko assumption. Also, theoretical analysis was performed to examine influence of the ratio of E_1 to G_{13} on optimum structure.

2 THEORETICAL OPTIMIZATION OF VARIABLE THICKNESS STRUCTURE

2.1 Theoretical analysis under Euler-Bernoulli Assumption

Analytical model was set as a cantilever, which was simplified from simple supported beam at both ends under concentrated load as shown in Figure 1. In this case, shear stress was ignored under Euler-Bernoulli assumption. This optimization problem can be expressed as obtaining variable thickness *t(x)* that minimizes total weight *W* subjected to same maximum deflection δ_{max}. The calculus of variations was applied to solve this problem theoretically.

Total weight *W* was calculated by the multiple of material density ρ, beam width *b*, beam length *l*, as shown in equation (1). By using the principle of virtual work, maximum deflection δ_{max} can be expressed as single integral equation, shown in equation (2). *M, P, E* and *I* are bending moment, applied concentrated load, elastic modulus and polar moment of inertia of cross-sectional area respectively. Minimizing *W* under same δ_{max} is equivalent to finding the extrema of functional *F*, shown in equation (3). λ is determined by boundary condition.

$$W = \rho b \int_0^l t(x)dx \tag{1}$$

$$\delta_{max} = \int_0^l \frac{M\bar{M}}{EI} dx = \frac{12P}{Eb} \int_0^l \frac{(l-x)^2}{t(x)^3} dx \tag{2}$$

$$F = W + \lambda\delta_{max} = \int_0^l \left(\rho b t(x) + \lambda \frac{12P}{Eb} \frac{(l-x)^2}{t(x)^3} \right) dx \tag{3}$$

The extrema can be obtained by solving the associated Euler–Lagrange equation, as shown in equation (4). *f* is an integrand of equation (3). *t(x)* was expressed as equation (5). As boundary condition, it was supposed that δ_{max} of variable thickness cantilever was equivalent to maximum deflection of a flat cantilever that has width *b*, length *l* and constant thickness t_1. Finally, optimum *t(x)* can be expressed only by constant thickness t_1 and beam length *l*, as shown in equation (6).

$$\frac{\partial f}{\partial t} - \frac{d}{dx}\left(\frac{\partial f}{\partial t'}\right) = 0 \tag{4}$$

$$t(x) = \left(\lambda \frac{36P}{\rho E b^2} \right)^{\frac{1}{4}} (l-x)^{\frac{1}{2}} \tag{5}$$

$$t(x) = 2^{\frac{1}{3}} t_1 \left(\frac{l-x}{l} \right)^{\frac{1}{2}} \tag{6}$$

Figure 2 shows comparison of flat and variable thickness cantilever with same maximum deflection. When t_1 is 10mm, optimum $t(x)|_{x=0}$ should be $10\sqrt[3]{2}$ (=12.6) mm. Total weight reduction was obtained by comparing flat and variable thickness cantilever; approximately 16.0%.

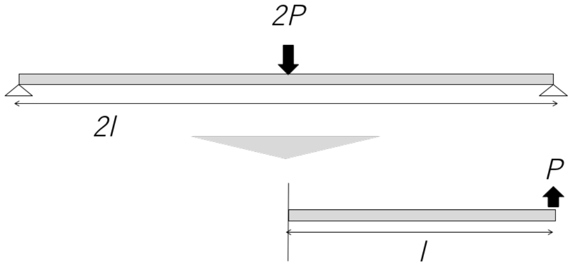

Figure 1: Schematics of analytical model

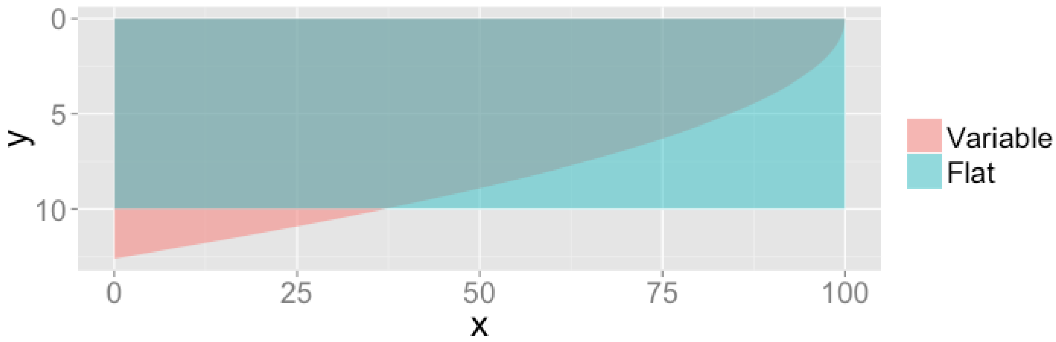

Figure 2: Comparison of variable and flat thickness cantilever with same maximum deflection

2.2 Verification by Finite element method

To verify the correctness of theoretical analysis, shape optimization was performed by using FE model. Free-shape optimization on 3D solid element was adopted instead of size optimization on 2D shell element to avoid discontinuous thickness between adjacent elements. As geometric condition, cantilever length *l*, width *b* and initial thickness t_0 were set as 100mm, 10mm and 10mm respectively. As boundary condition, concentrated load *P,* 100N, was applied at one edge. The other edge was fixed but can be moved to only z-direction in free-shape optimization process. Maximum deflection was set as 0.19mm, which was that of flat cantilever that thickness was 10mm. Altair® OptiStruct® was used for optimization solver.

Figure 3 shows schematics of analytical model and optimization result. Color contour represents shape change in z direction. As shown in optimization result, the side applied by concentrated load changed to be thinner and the other side changed to be thicker. As shown in Figure 4, the result of theoretical analysis was well consistent with that of FE optimization except at the edge applied by concentrated load. It is considered that difference at the edge was caused by the characteristics and fineness of FE elements.

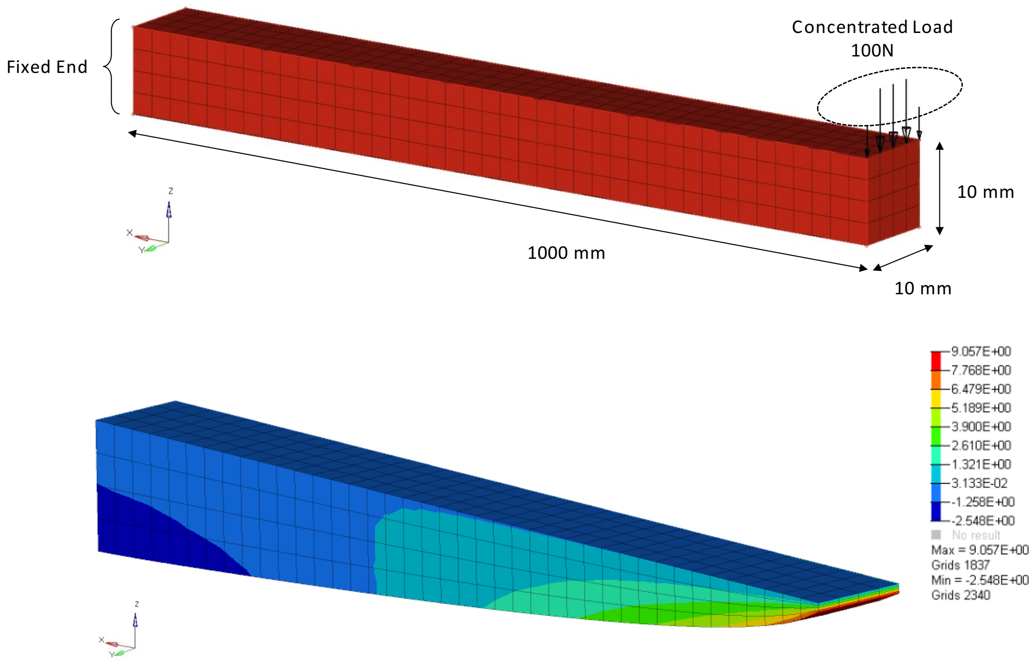

Figure 3: FE model of cantilever optimization

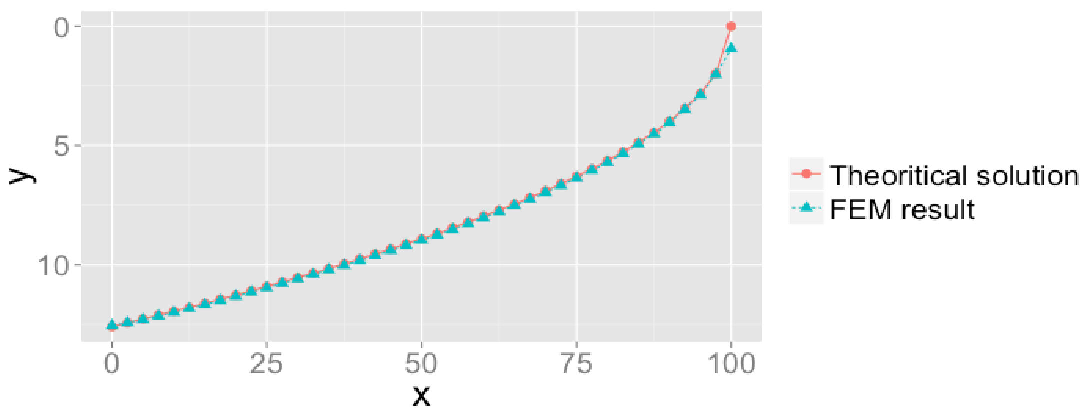

Figure 4: Comparison between theoretical solution and FEM result

3 THE INFLUENCE OF THE RATIO OF E_1 TO G_{13} ON OPTIMUM STRUCURE

3.1 Consideration of shear stress under Timoshenko assumption

As mentioned in Introduction, out-of-plane shear modulus of CFRTP is relatively lower. Therefore, the influence of shear stress cannot be neglected in actual case. Instead of Euler-Bernoulli assumption, Timoshenko assumption was applied in this section to examine the influence of the ratio of E_1 to G_{13} on optimum structure.

Equation (7) shows maximum deflection considering shear stress. k_T is a shear correction factor, given by Poisson's ratio ν based on Cowper's study [8], shown in equation (8). Functional F is expressed in equation (9) and boundary condition is given as same maximum deflection of flat cantilever in equation (10). However, theoretical solution cannot be obtained by solving the calculus of variations because impossible integrals are included in equation (10). Thus, numerical integration by Simpson's method was performed and λ that meets equation (10) was obtained by solving inverse problem.

$$\delta_{max} = \frac{12P}{Eb}\int_0^l \frac{(l-x)^2}{t(x)^3}\,dx + \frac{P}{k_T G b}\int_0^l \frac{1}{t(x)}dx \tag{7}$$

$$k_T = \frac{10(1+\nu)}{12+11\nu} \tag{8}$$

$$F = W + \lambda\delta_{max} = \int_0^l \left(\rho b t(x) + \lambda\frac{12P}{Eb}\frac{(l-x)^2}{t(x)^3} + \lambda\frac{P}{k_T G b}\frac{1}{t(x)}\right)dx \tag{9}$$

$$\frac{4l^3}{Et_1^3} + \frac{l}{Gk_T t_1} = \frac{12}{E}\int_0^l \frac{(l-x)^2}{t(x)^3}\,dx + \frac{1}{k_T G}\int_0^l \frac{1}{t(x)}dx \tag{10}$$

Figure 5 shows the optimum result under Timoshenko assumption and the comparison with flat thickness cantilever. Cantilever length l, width b and flat thickness t_l were set as 100mm, 10mm and 10mm respectively. In this case, CTT (chopped carbon fiber tape reinforced thermoplastics) was adopted for CFRTP material. Based on the experimental results of CTT/PP, E_l, G_{13} and ν were set as 34,000 MPa, 1340 MPa and 0.33 respectively [9-10]. CTT is quasi-isotropic in in-plane direction. As shown in Figure 5, general shape of optimum structure was similar to Figure 2, but minimum thickness was required under Timoshenko assumption to support shear stress at the edge. Due to this minimum thickness, total weight reduction under Timoshenko assumption was approximately 15.0%. This value was smaller than that under Euler-Bernoulli assumption.

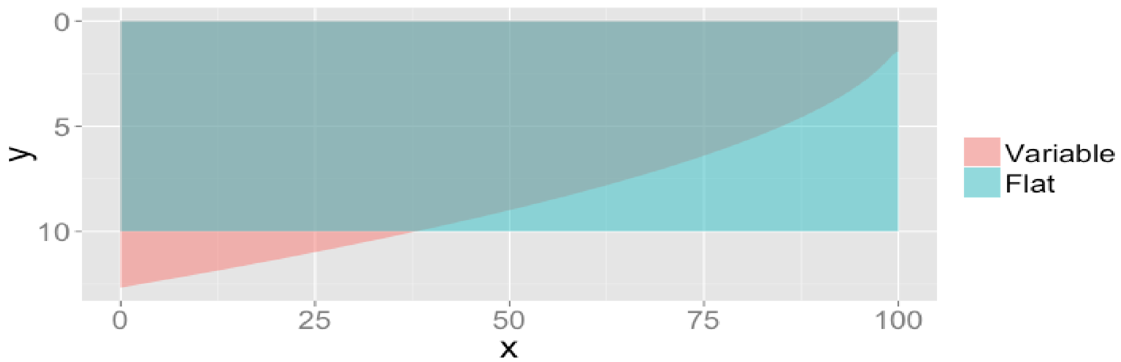

Figure 5: Comparison of flat and variable thickness cantilever with same maximum deflection considering shear stress

3.2 Sensitivity analysis to minimum thickness

Equation (11) was obtained by solving the calculus of variations of equation (9). In this case, *t(l)* means minimum thickness and *t (0)* does maximum thickness. Even though *t (0)* is also subjected to other parameters, it is shown that minimum thickness *t(l)* is almost proportional to a square root of E_1/G_{13}. In addition, equation (11) shows length l and shear correction factor k_T influences minimum thickness.

$$t(l) = t(0)\sqrt{\frac{1}{36\frac{G}{E}k_T\left(\frac{l}{t(0)}\right)^2 + 1}} \tag{11}$$

Figure 6 shows the relationship between minimum thickness and a square root of E_1/G_{13}. Dashed line means a square root of E_1/G_{13} of steel, aluminum, CTT and CMT (carbon fiber mat reinforced thermoplastics). The material properties of CMT also came from experimental data [9-10]. As shown in Figure 6, the ratio of E_1 to G_{13} of CFRTP materials, CTT and CMT, was higher than that of steel and aluminum. Thus, CTT and CMT require more minimum thickness at their edge. For example, when length l is 100mm, steel requires 0.45mm but CTT requires 1.45mm as minimum thickness at their edge; minimum thickness of CTT is three times as thick as that of steel.

It is also noteworthy that elastic modulus of CFRTP has temperature dependency. Figure 8 shows temperature influence on E_1 and G_{13} of CMT [6]. It is shown that both E_1 and G_{13} decreases as temperature increases, but G_{13} decreases more rapidly. Thus, the ratio of E_1 to G_{13} increases as temperature increases. In addition, it has been reported that mechanical properties of CTT vary as specimen thickness becomes thinner [11]. These characteristics mean CFRTP requires more minimum thickness at the edge to secure a sufficient safety factor, especially under high temperature circumstance.

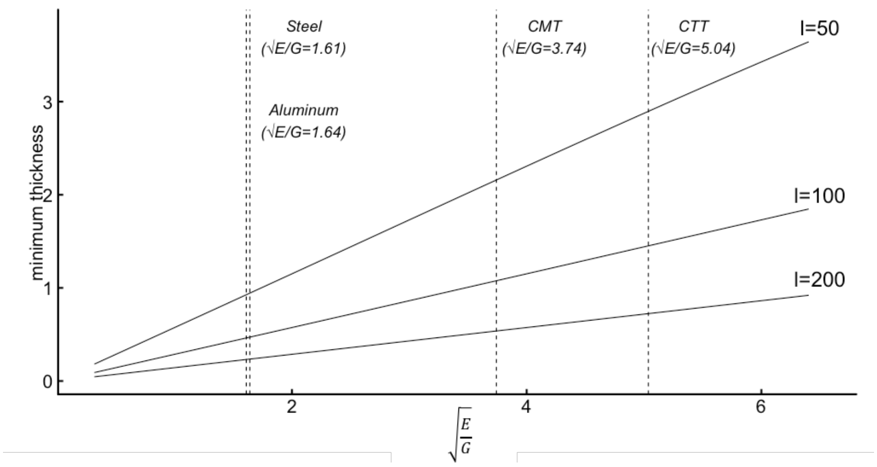

Figure 6: The relationship between minimum thickness and a square root of E_1/G_{13}

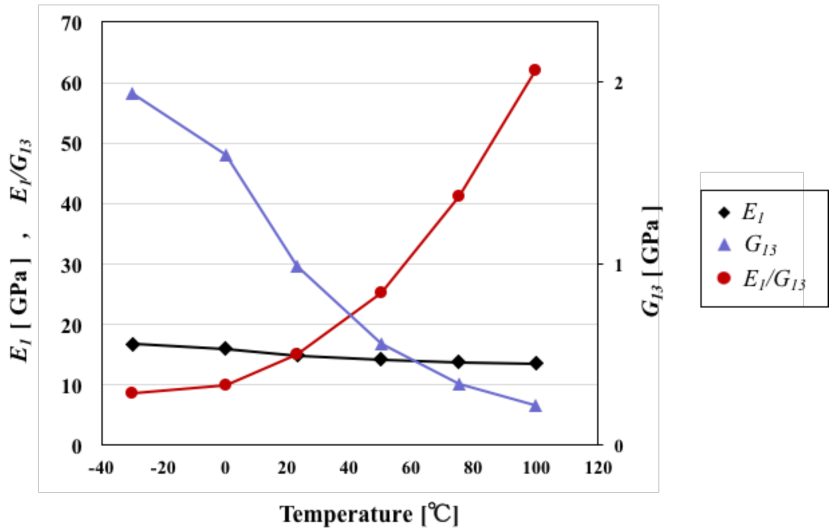

Figure 7: The temperature influence on E_1 and G_{13} of CMT (PP) [6]

4 CONCLUSIONS

In this study, optimum variable thickness of simple supported beam was theoretically obtained by solving the calculus of variations under Euler-Bernoulli and Timoshenko assumption. Also, theoretical analysis was performed to examine influence of the ratio E_1 to G_{13} on optimum structure. The following conclusions were obtained.

- Under Euler-Bernoulli assumption, theoretical solution of variable thickness structure can be obtained by solving the calculus of variations. Theoretical solution was well consistent with FE optimization, except for the supporting edge, where FE meshing fineness affected.
- Timoshenko assumption was applied to examine the effect of the ratio of E_1 to G_{13}. Although explicit analytical solution was not obtained under Timoshenko assumption, numerical solution was obtained by inverse problem analysis. In comparison with the result of Euler-Bernoulli assumption, it was observed that the minimum thickness was required on the edge under Timoshenko assumption to meet with shear stress. Through the parametric study, it was observed that this minimum thickness was almost proportional to a square root of E_1/G_{13}.
- In comparison with flat beam having same rigidity, total weight reduction of variable thickness beam was approximately 16.0% under Euler-Bernoulli assumption and 15.0% under Timoshenko assumption. This difference was also caused by minimum thickness.

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