ELASTIC WAVE PROPAGATION ANALYSIS IN A 2D FUNCTIONALLY GRADED HOLLOW CYLINDER USING SPECTRAL FINITE ELEMENT METHOD

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ABSTRACT

Functionally graded material (FGM) is a kind of special composite which components and material properties vary continuously along some directions. The material has been used in aerospace, nuclear, military and automotive industry fields because of its excellent designability and mechanical performance at the interface. In those applications, transient dynamic response analysis is an important problem, especially for structural health monitoring or impact load identification.

A two-dimensional axisymmetric element is proposed to simulate the wave propagation in functionally graded cylinders, which is established by time-domain spectral finite method (SFEM). The interpolation nodes of the element are collocated at the Gauss-Lobatto-Legendre (GLL) points to avoid the Runge phenomenon and get high accuracy. Besides, the mass matrix is diagonal due to the orthogonality of the approximation functions. Hence, the time-domain SFEM is more effective compared to the conventional FEM for wave propagation problems. In addition, the high-order shape functions used in SFEM can give better approximation of varying material properties. Therefore, the time-domain SFEM is particularly suitable for wave propagation analyzes of complex FGM structures.

Elastic wave propagation in a functionally graded cylinder subjected to an impulsive loading is studied detailed. The effect of material grading pattern is analyzed. The results demonstrate the effectiveness of the present method for the analysis of elastic wave propagation in functionally graded solids with axial symmetry and the material composition variation has an important effect on structural wave propagation behavior.

1 INTRODUCTION

Functionally graded material is a class of heterogeneous composite materials whose composition and performance vary in one or more directions. Compared with the conventional composites, the FGM has the following advantages: 1) FGM as an interface layer can improve bond strength among the incompatible materials. 2) FGM as a coating or interface layer can decrease residual stress and thermal stress. 3) FGM can eliminates stress singularity at the interface intersections and stress-free points. 4) FGM can enhance the connection strength and reduce the crack driving force. Hence, FGM is increasingly used in several important fields to sustain extreme loads.

Many scholars have studied dynamic response of FGM structures with analytical method [2-7]. An infinite FG plate subject to a point impact was analyzed by Sun et al. [2]. They used the high-order shear deformation plate theory (HSDT) and Hamilton principle to obtain approximate theoretical
dynamics solutions. Lefebvre et al. [3] obtained the dispersion and power flow solutions for a FGM plate using the extension of Legendre polynomial approach. Lamb wave propagation in a FGM plate was studied based on the power series technique by Cao et al. [4]. A semi-analytical method, which combines the state space method and the one-dimensional differential quadrature method, was developed to obtain the dynamic responses of FG circular plates by Nie et al. [5]. Stress wave propagation in FGMs was analyzed using a simplified one-dimensional model by Bruck [6]. Nie et al. [7] studied the free and forced vibration of a FGM annular sector plate with simply supported radial edges and arbitrary circular edges. Although these analytical works can give some insight into the elastic wave propagation in FGM structures, they are often limited to very simple boundary and loading conditions.

Many works can be found in literatures studying the problem in complex FG structures with numerical method. Chakraborty et al. focus on the wave behaviors in FG beam. The axisymmetric finite element model was applied to study the wave propagation in a thick hollow cylinder with finite length made of two-dimensional FGM subjected to impact internal pressure by Asgari et al. [11]. Although the above contributions prove that the traditional finite element method can effectively solve the problem of wave propagation in complex geometric boundaries of FG structures, the computational cost is very large.

the time-domain spectral finite element method (SFEM), which was first proposed by Patera [12] in fluid dynamics, has been extended to analyze the problems of structural wave propagation [13-19]. This method can be viewed as a special type of finite element methods and bears some similarity with the p-version FEM. The key idea of time-domain SFEM is the adoption of specific high-order shape functions. The time-domain SFEM is particularly suitable for the wave propagation analysis of complex FGM structures. However, the efficient time-domain SFEM have not been used extensively to study the problem of wave propagation in functionally graded materials, except in Ref. [20], a two-dimensional plane strain time-domain spectral finite element method was presented for numerical modeling of wave propagation in FGM.

2 THEORETICAL BACKGROUND

In a typical 1D functionally graded cylinder which material is continuously varied in the z-direction, the material varied from pure ceramic in the top surface to pure metal in the bottom surface or vice versa. In such case, the volume fraction of the ceramic is proposed as

\[ V_c = \left(\frac{z}{L}\right)^n \]

where \( n \) stand for compositional gradient exponent. Based on the rule of mixtures, the distribution of the material properties can be expressed as:

\[ P(z) = (P_c - P_m) \left(\frac{z}{L}\right)^n + P_m \]

The Young’s modulus, Poisson’s ration and density can be calculated by Eq. (2).

Two-dimensional FGMs are usually composed of three or four distinct material phases where one or two of them are ceramics and the others are metal alloy phases, and the volume fractions of the constituents vary in a predetermined composition profile. Considering a 2D-FG hollow cylinder which
section is shown in Fig. 1. The material in the cylinder varies in both radial direction and axial direction. The \( r_i \) and \( r_o \) represent the internal radius and external radius, respectively. The subscripts \( c_1, c_2, m_1 \) and \( m_2 \) stand for two different ceramic and metal. According to the homogenization scheme in Reddy, the relation among the volume fractions is proposed as

\[
V_{c1} + V_{c2} + V_{m1} + V_{m2} = 1
\]  

\[(3)\]

It is assumed that the internal wall is composed of two distinct ceramic and the external wall is composed of two distinct metal. The variation of the volume factions of every component can be defined by the well-known power law:

\[
V_{c1} = \left[ 1 - \left( \frac{r-r_i}{r_o-r_i} \right)^{n_r} \right] \left[ 1 - \left( \frac{z}{L} \right)^{n_z} \right]
\]  

\[(4)\]

\[
V_{c2} = \left[ 1 - \left( \frac{r-r_i}{r_o-r_i} \right)^{n_r} \right] \left[ \left( \frac{z}{L} \right)^{n_z} \right]
\]  

\[(5)\]

\[
V_{m1} = \left[ \left( \frac{r-r_i}{r_o-r_i} \right)^{n_r} \right] \left[ 1 - \left( \frac{z}{L} \right)^{n_z} \right]
\]  

\[(6)\]

\[
V_{m2} = \left[ \left( \frac{r-r_i}{r_o-r_i} \right)^{n_r} \right] \left[ \left( \frac{z}{L} \right)^{n_z} \right]
\]  

\[(7)\]

where \( n_r \) and \( n_z \) represent the compositional gradient exponent in \( r \)-direction and \( z \)-direction, respectively. In the case of \( r_i = 1 \text{m}, r_o = 1.5 \text{m} \) and \( L = 1 \text{m} \), the distribution of the volume fraction \( V_{c2} \) and \( V_{m1} \) are illustrated in Fig. 2 and 3 for \( n_r = 2, n_z = 2 \).
Figure 3 The distribution of $V_{m1}$

Hence, the volume fractions of every constituent at arbitrary point in the 2D-FG cylinder can be calculated by Eqs. (4-7). Using the classical rule of mixture, the distribution of the material properties can be expressed as:

$$P = P_{c1}V_{c1} + P_{c2}V_{c2} + P_{m1}V_{m1} + P_{m2}V_{m2} \quad (8)$$

where $P$ is the material properties such as the Young’s modulus, the Poisson’s ration and the mass density $\rho$.

3 NUMERICAL RESULTS AND DISCUSSION

3.1 Model and analysis description

Elastic wave propagation in a two-dimensional functionally graded thick hollow cylinder shown in Figure 4 is studied. The dimensions of the cylinder are considered as thickness $h = 1$ m, internal radius $r_i = 1$ m and external radius $r_o = 1.5$ m. The inner wall of the cylinder is composed of two distinct ceramic and the outer wall is metal surface (two distinct alloy). The material properties of each component are listed in the Table 1. Power law is used to characterize the graded varying material composition. The cylinder is subjected to an internal pressure and the excitation signal is simulated by a product of a Hanning window multiplied by a sinusoidal signal with frequency 100 kHz, expressed by

$$P(t) = \frac{1}{2} P_0 \times (1 - \cos(\frac{2\pi f}{n}t)) \sin(2\pi ft) \quad (9)$$

In the case of typical values of $P_0 = 10^5$ pa, $n = 5$, the internal pressure is plotted in the Fig. 5.

<table>
<thead>
<tr>
<th>Constituents</th>
<th>Material</th>
<th>Young’s modulus E ($10^9$ pa)</th>
<th>Density $\rho$ (kg/m³)</th>
<th>Poisson’s Ratio $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>m1</td>
<td>Ti6Al4V</td>
<td>115</td>
<td>4515</td>
<td>0.31</td>
</tr>
<tr>
<td>m2</td>
<td>Al 1,100</td>
<td>69</td>
<td>2715</td>
<td>0.30</td>
</tr>
<tr>
<td>c1</td>
<td>SiC</td>
<td>440</td>
<td>3210</td>
<td>0.24</td>
</tr>
<tr>
<td>c2</td>
<td>Al2O3</td>
<td>300</td>
<td>3470</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Table 1 Basic constituents of the 2D-FG cylinder
3.2 Results and discussion

After a convergence analysis, it is found that using 6th order spectral element and meshing the structure with 3 elements in the radial direction and 9 elements in the axial direction (totaling 2090 DOF) can result a reasonable balance between computational accuracy and costs. Meanwhile, if the structure is analyzed by the conventional finite element method, it needs 5556 elements (totaling 12100 DOF) at least to obtain a reasonable accuracy. The time step of integration is chosen as $1 \times 10^{-7}$ s to satisfy the stable condition. To investigate the influence of the power law exponents on wave propagation, three material compositions, $n_r = n_z = 10$, $n_r = n_z = 1$, $n_r = n_z = 0.1$, were analyzed. Figures 6 illustrates the distribution of the displacement components $u(r, z)$ at $t = 150\mu s$ for $n_r = n_z = 0.1$, $n_r = n_z = 1$, $n_r = n_z = 10$, respectively. The case $n_r = n_z = 10$ stands for the ceramic-rich condition (especially $c1$ component) and $n_r = n_z = 0.1$ represents the metal-rich composition (especially $m2$ component). The last $n_r = n_z = 1$ represents a linearly uniform variation in both radial and axial directions.

In the case of $n_r = n_z = 0.1$, two wave packets are captured. The first distinct wave packet approximately located in the middle of the 2D-FG cylinder in the negative direction and the other packet just occurs in the edge of the inner wall in positive direction. The difference of the sign is caused by the variation of the excitation signal. Similarly, for $n_r = n_z = 1$ and $n_r = n_z = 10$, the two wave packets are shown in the figure, which one of them is in the positive direction and the other is negative. The first wave packet reaches the edge of the outer wall for $n_r = n_z = 1$ and gets compressed for $n_r = n_z = 10$. Therefore, it can be found that the elastic wave speed is faster as compositional gradient exponent $n_r$ and $n_z$ increase. To study the wave response time-histories, two evaluation points, A(1.125,0.8) and B(1.375,0.2), are chosen to output the time histories of displacement and stress component. As shown in Figure 7, the results of different gradient exponent are distinct. Thus, the gradient exponents have an evident effect on wave speed and peak value.
Figure 6. Distribution of the radial displacement component $u(r, z)$ (unit: m) $t = 150\mu s$

A(1.125,0.8)  
B(1.375,0.2)

Figure 7 Time histories of the displacement at evaluation points

A(1.125,0.8)  
B(1.375,0.2)
ACKNOWLEDGMENT

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REFERENCES


