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A SYSTEMATIC APPROACH TO INVARIANT-BASED ANISOTROPIC NON-ASSOCIATIVE PLASTICITY FOR FIBER REINFORCED POLYMERS

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ABSTRACT

In this contribution a review of several aspects of formulating yield surfaces and plastic potentials within the framework of isotropic tensor functions is presented. Assuming transversal isotropic material symmetry, as e.g. suitable for a wide range of fiber reinforced composites, general formulations for the yield surface and plastic potential are given in terms of the full quadratic form. Procedures to relate the set of parameters arising from the quadratic form to the material behaviour are outlined. Thereby, emphasis is put on the implementation of constitutive modelling assumptions that lead to specific formulations, obtained from the quadratic form. The considered modelling assumptions are: independence of the yield surface from longitudinal (fiber) stress and independence of hydrostatic stress, as well as incompressibility and inextensibility for the plastic potential.

1 INTRODUCTION

Constitutive modelling based on isotropic tensor functions is widely used for various classes of anisotropy for both, elastic and inelastic material behavior. For instance, it is employed to model non-linear elasticity of biological tissues [4] and the inelastic behavior of composites [10, 12]. The present contribution is concerned with the modelling of inelastic deformations for short and endless fiber reinforced composites and aims to demonstrate a general and systematic invariant-based modelling approach for the inelastic behavior of short or long fiber reinforced composites. The inelastic behaviour of some material is determined by the choice of two functionals: the yield surface that delimits the elastic region in stress space and the plastic potential that determines the direction of the plastic flow. In general, these functionals are given as tensor polynomials of order two, i.e. quadratic forms. This gives rise to several constitutive constants to be obtained from independent tests. However, in practice test data is not always available or feasible and, therefore, some constants are eliminated by making reasonable assumptions regarding the materials behaviour. These assumptions are to be respected in the formulation of the functionals, which often leads to particular specializations. In the present contribution several procedures to this end are reviewed.

2 ISOTROPIC TENSOR POLYNOMIALS

The following notation scheme is adopted here: Tensors of rank 1 are labeled by an underscore and tensors of rank 2 by a underscript tilde. Let $\underline{e}_i$ denote unit vectors aligned with the orthonormal coordinate axis $i = 1,2,3$. Furthermore, let $\underset{\sim}{\sigma} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j$ and $\underline{\sigma}^{\mathsf{T}} = \left[\begin{array}{cccccc} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13} \end{array} \right]$ denote the stress tensor and the stress components in Voigt's notation, respectively.

We indent to formulate scalar-valued, polynomial functions of tensorial arguments $\mathcal{I}(\underset{\sim}{M}_1,\ldots,\underset{\sim}{M}_n)$ to describe the constitutive behaviour of some material. Such functions must satisfy the principle of coordinate frame indifference, $\mathcal{I}(\underset{\sim}{M}_1,\ldots,\underset{\sim}{M}_n)=\mathcal{I}(\underset{\sim}{Q}\underset{\sim}{M}_1\underset{\sim}{Q}^{\mathsf{T}},\ldots,\underset{\sim}{Q}\underset{\sim}{M}_n\underset{\sim}{Q}^{\mathsf{T}})$, for any orthogonal transformation $\underset{\sim}{Q}$. If the function $\mathcal{I}$ complies with the previous statement it is called an isotropic tensor function [9]. An isotropic, scalar-valued tensor function suitable for isotropic materials is obtained if it is formulated exclusively in terms of so called basic invariants Υ_i [1, 5] calculated from the arguments $\underset{\sim}{M}_1,\ldots,\underset{\sim}{M}_n$ or combinations thereof. Tables for basic invariants for a number of combinations of arguments are available in the literature [1].

An anisotropic material with a certain class of symmetry that is characterized by direction vectors $\underline{a}_1,\ldots,\underline{a}_m$ can be associated with a group that preserves this symmetry $G=\{\underset{\sim}{Q}\subset O,\ \underset{\sim}{Q}\underline{a}_1=\underline{a}_1,\ldots,\ \underset{\sim}{Q}\underline{a}_m=\underline{a}_m\}$, where $\underset{\sim}{Q}$ now is a member of a proper subset of the complete orthogonal group O. In order to obtain an isotropic tensor function for such an anisotropic material an isotropization theorem can be invoked [5, 13]. It states that for an anisotropic material a representation in terms of an isotropic tensor function is obtained by including the direction vectors in the list of arguments and formulating the basic invariants Υ_i according to this extended argument list.

In the present case, transversally isotropic materials, that is materials characterised by the symmetry group $G=\{\underset{\sim}{Q}\in O,\ \underset{\sim}{Q}\underline{a}=\underline{a}\}$, are considered. Besides the single direction vector $\underline{a}$, only the stress tensor $\underset{\sim}{\sigma}$ is admitted as an argument to the constitutional functions considered here. In the following, the symmetry of the material is represented by the so-called structural tensor $\underset{\sim}{A}=\underline{a}\otimes\underline{a}$. The structural tensor satisfies $\underset{\sim}{A}=\underset{\sim}{Q}\underset{\sim}{A}\underset{\sim}{Q}^{\mathsf{T}}$ for all $\underset{\sim}{Q}$ in the symmetry group G. Without loss of generality, one can identify the preferential direction, i.e. the fiber direction, with the first coordinate direction, therefore:

$$\underline{a}=\begin{bmatrix}1\\0\\0\end{bmatrix}\qquad\qquad \underset{\sim}{A}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \tag{1}$$

The 10 basic invariants $\Upsilon_{1..10}$ for the two symmetric tensorial arguments in $\mathcal{I}(\underset{\sim}{\sigma},\underset{\sim}{A})$ are [5]:

$$\begin{array}{cccc} & \operatorname{tr}(\underset{\sim}{\sigma}) & \operatorname{tr}(\underset{\sim}{\sigma}^2) & \operatorname{tr}(\underset{\sim}{\sigma}^3)\\ & \operatorname{tr}(\underset{\sim}{A}) & \operatorname{tr}(\underset{\sim}{A}^2) & \operatorname{tr}(\underset{\sim}{A}^3)\\ \operatorname{tr}(\underset{\sim}{\sigma}\underset{\sim}{A}) & \operatorname{tr}(\underset{\sim}{\sigma}\underset{\sim}{A}^2) & \operatorname{tr}(\underset{\sim}{\sigma}^2\underset{\sim}{A}) & \operatorname{tr}(\underset{\sim}{\sigma}^2\underset{\sim}{A}^2)\end{array} \tag{2}$$

Since only 2$^{\text{nd}}$-order polynomials are considered here, the basic invariant $\operatorname{tr}(\underset{\sim}{\sigma}^3)$ in the first row of (2) is excluded. The equality $\underset{\sim}{A}=\underset{\sim}{A}^2=\underset{\sim}{A}^3=\text{const.}$ renders all three basic invariants in the second row, plus two more in the third row, redundant. The remaining 4 basic invariants are recombined to form new invariant expressions, more pertinent for the present purpose. Following Hashin [3] the following set of invariants is adopted:

$$\begin{aligned} I_1(\underset{\sim}{\sigma},\underset{\sim}{A}) &= +\tfrac{1}{2}\operatorname{tr}(\underset{\sim}{\sigma}^2)-\operatorname{tr}(\underset{\sim}{\sigma}^2\underset{\sim}{A})+\operatorname{tr}(\underset{\sim}{\sigma}\underset{\sim}{A})\operatorname{tr}(\underset{\sim}{\sigma})-\tfrac{1}{2}\operatorname{tr}(\underset{\sim}{\sigma})^2 &&= \sigma_{23}^2-\sigma_{22}\sigma_{33} && (3_1)\\ I_2(\underset{\sim}{\sigma},\underset{\sim}{A}) &= -\operatorname{tr}(\underset{\sim}{\sigma}\underset{\sim}{A})^2+\operatorname{tr}(\underset{\sim}{\sigma}^2\underset{\sim}{A}) &&= \sigma_{12}^2+\sigma_{13}^2 && (3_2)\\ I_3(\underset{\sim}{\sigma},\underset{\sim}{A}) &= +\operatorname{tr}(\underset{\sim}{\sigma}\underset{\sim}{A}) &&= \sigma_{11} && (3_3)\\ I_4(\underset{\sim}{\sigma},\underset{\sim}{A}) &= -\operatorname{tr}(\underset{\sim}{\sigma}\underset{\sim}{A})+\operatorname{tr}(\underset{\sim}{\sigma}) &&= \sigma_{22}+\sigma_{33} && (3_4)\end{aligned}$$

Hashin's choice for the reformulated invariants has the advantage over the basic invariants that they can be associated with particular stress states: I_2 and I_3 are clearly related to longitudinal-transversal shear and longitudinal (fiber) normal stress. The Invariants I_1 and I_4 on the other hand relate to purely transversal properties, related to size and center of Mohr's circle for the transversal plane, respectively. This substantially simplifies parameter identification.

3 INVARIANT REPRESENTATION OF 2ND-ORDER TENSOR POLYNOMIALS

To formulate scalar-valued, coordinate frame invariant tensor polynomials of order 2 in the stress components the invariants from (3) are separated into 1st- and 2nd-order terms.

1st-order:

$$I_3 = \sigma_{11} \qquad I_4 = \sigma_{22} + \sigma_{33} \tag{4}$$

2nd-order:

$$\begin{aligned} I_1 &= \sigma_{23}^2 - \sigma_{22}\sigma_{33} & I_2 &= \sigma_{12}^2 + \sigma_{13}^2 \\ I_3^2 &= \sigma_{11}^2 & I_4^2 &= (\sigma_{22} + \sigma_{33})^2 \\ I_3 I_4 &= \sigma_{11}(\sigma_{22} + \sigma_{33}) \end{aligned} \tag{5}$$

Quadratic invariants I_1 and I_2 are included as powers of 1 only, linear invariants I_3 and I_4 are included as powers of 1 and 2. In total, this renders 7 invariant terms up to order 2 in the stress components. Each term is associated with a constitutive coefficient $\iota_{i(j)}$. Indices $i(,j)$ denote (repeated) multiplication with the invariant(s) of the corresponding number. In general, an additive constant ι_0 might also be present, raising the number of independent constants to 8. A generic scalar-valued tensor polynomial $\mathcal{I}$, invariant under transformations that do not change $\underset{\sim}{\boldsymbol{A}}$, can be written in the following form:

$$\mathcal{I}(\underline{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}) = \iota_0 + \iota_1 I_1 + \iota_2 I_2 + \iota_3 I_3 + \iota_4 I_4 + \iota_{33} I_3^2 + \iota_{34} I_3 I_4 + \iota_{44} I_4^2 \tag{6}$$

For an adaption of (6) for a failure surface cf. e.g. [3, equation (6)]. To ease manipulation, the following compact form, containing the coefficients ι_{ij} in matrix components, is used:

$$\mathcal{I}(\underline{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}) = \underline{\boldsymbol{\sigma}}^{\mathsf{T}} \underline{\underline{\boldsymbol{\mathit{I\!I}}}}(\underset{\sim}{\boldsymbol{A}})\, \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{I}}(\underset{\sim}{\boldsymbol{A}})\, \underline{\boldsymbol{\sigma}} + \iota_0 \tag{7}$$

There, $\underline{\underline{\boldsymbol{\mathit{I\!I}}}}$ is the Hessian and $\underline{\boldsymbol{I}}$ is the gradient at the origin, both depend on $\underset{\sim}{\boldsymbol{A}}$. They remain invariant under transformations that leave $\underset{\sim}{\boldsymbol{A}}$ invariant:

$$\underline{\underline{\boldsymbol{\mathit{I\!I}}}}(\underset{\sim}{\boldsymbol{A}}) = \frac{1}{2} \frac{\partial^2}{\partial \underline{\boldsymbol{\sigma}} \partial \underline{\boldsymbol{\sigma}}} \Big(\mathcal{I}(\underline{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}) \Big) \qquad \underline{\boldsymbol{I}}(\underset{\sim}{\boldsymbol{A}}) = \frac{\partial}{\partial \underline{\boldsymbol{\sigma}}} \Big(\mathcal{I}(\underline{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}) \Big) \Big|_{\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{0}}} \tag{8}$$

A good point of reference for a demonstration is linear elasticity. Consider e.g. the complementary, i.e. Legendre transformed, strain potential Ψ^* in (9), where $\underset{\sim}{\boldsymbol{S}}(\underset{\sim}{\boldsymbol{A}})$ represents the compliance and $\underline{\boldsymbol{\alpha}}(\underset{\sim}{\boldsymbol{A}})\,\Delta T$ the thermal strain.

$$\Psi^* = \frac{1}{2} \underline{\boldsymbol{\sigma}}^{\mathsf{T}} \underset{\sim}{\boldsymbol{S}}(\underset{\sim}{\boldsymbol{A}})\, \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{\sigma}}^{\mathsf{T}} \underline{\boldsymbol{\alpha}}(\underset{\sim}{\boldsymbol{A}})\, \Delta T \tag{9}$$

The coefficients of the 2nd-order terms contribute to the compliance matrix, while the coefficients of 1st-order terms result in a shift of the minimum from the origin caused by thermal strain. It is well known [2] that for transversal isotropy there are 5 independent constants in the stiffness-, respectively compliance-, matrix and 2 more in the vector of the thermal coefficients adding up to a total of 7. The eighth, constant coefficient is an inconsequential additive constant to Ψ^* in this case and is omitted. If the structural tensor vanishes, $\underset{\sim}{\boldsymbol{S}}(\underset{\sim}{\boldsymbol{0}})$ and $\underline{\boldsymbol{\alpha}}(\underset{\sim}{\boldsymbol{0}})$ assume the isotropic form with 2 and 1 independent constants, respectively.

4 YIELD FUNCTIONS

Typical material models for fiber reinforced materials are subject to several constitutive constraints and/or simplifying model assumptions regarding their constitutive behaviour. The implementation of such constitutive constraints relies on suitable, linear decompositions of the stress tensor to isolate the components which are to be excluded in the formulation of the yield surface. This is in line with the principles stated in Section 3, because the invariants used to formulate the yield surface can be linear combinations of the basic invariants in (2). To this effect, stress components which are thought not to contribute to the yielding are isolated in a stress tensor $\underset{\sim}{\boldsymbol{\sigma}}^r$, the reactive stress. To obtain the plasticity inducing stress, $\underset{\sim}{\boldsymbol{\sigma}}^p = \underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^r$, the reactive stress is subtracted from the total stress.

To provide an example, the set of assumptions, common for brittle endless fibers reinforcing a somewhat ductile matrix, shall be considered. For this kind of material it is typically assumed that brittle fiber failure dominates the longitudinal direction, therefore the projection of stress onto the fiber direction is not to be considered in the definition of the yield surface (i). Furthermore, it shall be enforced, that yielding does not occur under purely hydrostatic stress (ii). In this case a suitable decomposition is given by (10) [6, 7]:

$$\underset{\sim}{\boldsymbol{\sigma}}^r = \frac{1}{2}\big(\operatorname{tr}(\underset{\sim}{\boldsymbol{\sigma}}) - \operatorname{tr}(\underset{\sim}{\boldsymbol{\sigma}}\underset{\sim}{\boldsymbol{A}})\big)\underset{\sim}{\mathbf{1}} + \frac{1}{2}\big(\operatorname{tr}(\underset{\sim}{\boldsymbol{\sigma}}) - 3\operatorname{tr}(\underset{\sim}{\boldsymbol{\sigma}}\underset{\sim}{\boldsymbol{A}})\big)\underset{\sim}{\boldsymbol{A}} \tag{10}$$

From this point on, there are two possible ways to proceed constructing the yield surface as is elaborated below.

It seems natural to formulate the yield surfaces in terms of the plasticity inducing part of the stress only, rather the total stress. Writing out the invariants (3) with the plasticity inducing stress $\underset{\sim}{\boldsymbol{\sigma}}^p$ substituting the total stress $\underset{\sim}{\boldsymbol{\sigma}}$ and the decomposition (10) reveals that $I_3(\underset{\sim}{\boldsymbol{\sigma}}^p, \underset{\sim}{\boldsymbol{A}})$ and $I_4(\underset{\sim}{\boldsymbol{\sigma}}^p, \underset{\sim}{\boldsymbol{A}})$ vanish identically. Thus, the constraint is intrinsically incorporated into the yield surface, (11).

$$\hat{\mathcal{F}}(\underset{\sim}{\boldsymbol{\sigma}}^p, \underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) = -1 + \hat{\alpha}_1 I_1(\underset{\sim}{\boldsymbol{\sigma}}^p, \underset{\sim}{\boldsymbol{A}}) + \hat{\alpha}_2 I_2(\underset{\sim}{\boldsymbol{\sigma}}^p, \underset{\sim}{\boldsymbol{A}}) \tag{11}$$

Even though, the procedure is straight forward for the constitutive constraints considered above, some inconsistency can arise if, to better match experimental results, further effects are to be considered in the yield function. For instance, in [12] the form (11) was supplemented by additional invariants formulated with respect to the total stress $\underset{\sim}{\boldsymbol{\sigma}}$ rather than the only the plasticity inducing part. This was done in order to introduce the tensile compressive asymmetry typically observed for fiber reinforced composites. A consequence of the inclusion of extra invariants with respect to total stress is that (10) is no longer purely reactive, causing some inconsistency.

The approach discussed above imposes constitutive constraints directly via the invariants in the yield function and can, thus, be seen as 'bottom up' modelling procedure. The converse approach is to start with the yield surface in the most general form for the given material symmetry class and incorporate constraints 'top down' during parameter identification. In this spirit, the yield function is defined as a generic polynomial of order 2 in the manner specified by (6). From there, the generic polynomial is renamed to $\mathcal{F}$, the yield function, and the constitutive constants to α_{ij}. The constant α_0 is normalized to -1. Aiming for a hardening model with a single, scalar internal variable, the remaining constitutive coefficients are functions of an suitably defined equivalent plastic strain $\bar{\varepsilon}$. For clarity, the arguments $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{\boldsymbol{A}}$ of the invariants and the argument $\bar{\varepsilon}$ of α_{ij} is omitted in print. Thus, the generic form of a yield surface is obtained in terms of 7 invariant terms with a corresponding amount of coefficients:

$$\mathcal{F}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) = -1 + \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4 + \alpha_{33} I_3^2 + \alpha_{34} I_3 I_4 + \alpha_{44} I_4^2 \tag{12}$$

The compact notation (13), with the Hessian $\underset{\sim}{\boldsymbol{K}}$ and gradient $\underline{\boldsymbol{L}}$ according to (14), makes apparent that

the general yield function (12) corresponds to the classical formulation for yield, respectively failure, surfaces in [8, 11].

$$\mathcal{F}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) = \underline{\boldsymbol{\sigma}}^{\mathsf{T}} \underset{\sim}{\boldsymbol{K}}(\underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{L}}(\underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon})^{\mathsf{T}} \underline{\boldsymbol{\sigma}} - 1 \tag{13}$$

$$\underset{\sim}{\boldsymbol{K}}(\underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) = \begin{bmatrix} \alpha_{33} & \frac{1}{2}\alpha_{34} & \frac{1}{2}\alpha_{34} & 0 & 0 & 0 \\ \frac{1}{2}\alpha_{34} & \alpha_{44} & \alpha_{44} - \frac{1}{2}\alpha_1 & 0 & 0 & 0 \\ \frac{1}{2}\alpha_{34} & \alpha_{44} - \frac{1}{2}\alpha_1 & \alpha_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 \end{bmatrix} \qquad \underline{\boldsymbol{L}}(\underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) = \begin{bmatrix} \alpha_3 \\ \alpha_4 \\ \alpha_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

If a constraint along the lines of (10) is present, the coefficients must satisfy certain relations so that the reactive part of the total stress does not contribute to $\mathcal{F}$. These can be found from the requirement below, which must hold for any stress state. For the form of the reactive stress specified in (10) one finds from (15) the condition (16) under which (12) reverts to (11):

$$(\underline{\boldsymbol{\sigma}}^r)^{\mathsf{T}} \underset{\sim}{\boldsymbol{K}} \underline{\boldsymbol{\sigma}}^r + \underline{\boldsymbol{L}}^{\mathsf{T}} \underline{\boldsymbol{\sigma}}^r = 0 \tag{15}$$

$$\alpha_3 = \alpha_4 = \alpha_{33} = \alpha_{34} = 0 \qquad\qquad \alpha_1 = 4\alpha_{44} \tag{16}$$

In absence of any constraints or modelling assumptions of the kind of (10) the coefficients $\alpha_{i(j)}$ are to be determined from 7 independent calibration experiments. Since they control the size and shape of the elastic region as a function of the equivalent plastic strain variable $\bar{\varepsilon}$ they can be related to yield strength curves, $y^\tau(\bar{\varepsilon})$, of some material tests τ. Following the procedure laid out in [11], a possible choice of uni/bi-axial and shear stress states from which all coefficients $\alpha_{i(j)}$ can be determined are given in (17). There, and in the following, the argument $\bar{\varepsilon}$ of y^τ is omitted. The abbreviations in the superscript stand for: transverse/in-plane shear (ts/is), fiber tension/compression (ft/fc), transverse tension/compression (tt/tc) and biaxial (bi).

$$\begin{aligned}
\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{ts}}(\bar{\varepsilon}) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & y^{\mathrm{ts}} \\ 0 & y^{\mathrm{ts}} & 0 \end{bmatrix} & \underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{ft}}(\bar{\varepsilon}) &= \begin{bmatrix} +y^{\mathrm{ft}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{tt}}(\bar{\varepsilon}) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & +y^{\mathrm{tt}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{is}}(\bar{\varepsilon}) &= \begin{bmatrix} 0 & y^{\mathrm{is}} & 0 \\ y^{\mathrm{is}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{fc}}(\bar{\varepsilon}) &= \begin{bmatrix} -y^{\mathrm{fc}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{tc}}(\bar{\varepsilon}) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -y^{\mathrm{tc}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{bi}}(\bar{\varepsilon}) &= \begin{bmatrix} y^{\mathrm{bi}} & 0 & 0 \\ 0 & y^{\mathrm{bi}} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{17}$$

From $\mathcal{F}(\underset{\sim}{\boldsymbol{\sigma}}^\tau, \underset{\sim}{\boldsymbol{A}}, \bar{\varepsilon}) = 0$ for every test τ one finds (cf. [11]):

$$\begin{aligned}
\alpha_1(\bar{\varepsilon}) &= \frac{1}{(y^{\mathrm{ts}})^2} & \alpha_3(\bar{\varepsilon}) &= \frac{1}{y^{\mathrm{ft}}} - \frac{1}{y^{\mathrm{fc}}} & \alpha_{33}(\bar{\varepsilon}) &= \frac{1}{y^{\mathrm{fc}} y^{\mathrm{ft}}} \\
\alpha_2(\bar{\varepsilon}) &= \frac{1}{(y^{\mathrm{is}})^2} & \alpha_4(\bar{\varepsilon}) &= \frac{1}{y^{\mathrm{tt}}} - \frac{1}{y^{\mathrm{tc}}} & \alpha_{44}(\bar{\varepsilon}) &= \frac{1}{y^{\mathrm{tc}} y^{\mathrm{tt}}}
\end{aligned} \tag{18}$$

$$\alpha_{34}(\bar{\varepsilon}) = \frac{1}{(y^{\mathrm{bi}})^2} + \frac{1}{y^{\mathrm{bi}}}\left(\frac{1}{y^{\mathrm{fc}}} - \frac{1}{y^{\mathrm{ft}}} + \frac{1}{y^{\mathrm{tc}}} - \frac{1}{y^{\mathrm{tt}}}\right) - \frac{1}{y^{\mathrm{fc}}y^{\mathrm{ft}}} - \frac{1}{y^{\mathrm{tc}}y^{\mathrm{tt}}}$$

5 PLASTIC POTENTIAL

Definition of a plastic potential $\mathcal{G}$ separate from $\mathcal{F}$ allows fitting of experimental data for plastic strain ratios, i.e. plastic Poisson's ratios, independently of the yield stress data. For instance, it allows to maintain plastic incompressibility while considering the effect of hydrostatic pressure on the yield stresses. This means that, in general, a second set of constitutive constraints/assumptions is present for the plastic potential and the same 'top down' approach that was motivated for the yield function is applicable to plastic potentials as well.

The initial form is, again, obtained from (6) by renaming the constitutive constants to $\beta_{i(,j)}$. However, to account for the circumstance that experimental data for plastic strain ratios rarely distinguishes between tension and compression there is little need for the linear terms which are, therefore, outright omitted here. Furthermore, for simplicity's sake, no evolution of the plastic potential with respect to internal variables shall be considered. The resulting form is (19) with 5 independent parameters:

$$\mathcal{G}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{A}}) = -1 + \beta_1 I_1 + \beta_2 I_2 + \beta_{33} I_3^2 + \beta_{34} I_3 I_4 + \beta_{44} I_4^2 \tag{19}$$

Via the evolution equation for the plastic strain, $\dot{\underset{\sim}{\varepsilon}}^p = \lambda \partial \mathcal{G} / \partial \underset{\sim}{\boldsymbol{\sigma}}$, this leads to the following plastic strain rate:

$$\dot{\underset{\sim}{\varepsilon}}^p(\underset{\sim}{\boldsymbol{\sigma}}) = \lambda \begin{bmatrix} 2\beta_{33}\sigma_{11} + \beta_{34}(\sigma_{22}+\sigma_{33}) & 2\beta_2\sigma_{12} & 2\beta_2\sigma_{13} \\ 2\beta_2\sigma_{12} & \beta_{34}\sigma_{11} - \beta_1\sigma_{33} + 2\beta_{44}(\sigma_{22}+\sigma_{33}) & 2\beta_1\sigma_{23} \\ 2\beta_2\sigma_{13} & 2\beta_1\sigma_{23} & \beta_{34}\sigma_{11} - \beta_1\sigma_{22} + 2\beta_{44}(\sigma_{22}+\sigma_{33}) \end{bmatrix} \tag{20}$$

Due to the arbitrary scaling by the plastic consistency factor λ it is apparent that the absolute magnitude of the coefficients $\beta_{i,(j)}$ is not important and only their ratios matter, i.e. the size of the plastic potential does not affect the resulting strain rate. Furthermore, it is apparent that β_2 is completely decoupled from the rest of the constants. It seems reasonable to ignore β_2 and calculate the in-plane plastic shear rates from an associated flow rule $\dot{\varepsilon}^p_{12} = \lambda \partial \mathcal{F} / \sigma_{12}$.

To identify the remaining three parameters test cases need to be devised and, for this purpose, the plastic strain ratios for uniaxial load cases are considered (21). It has already been stated that no distinction for tensile and compressive behaviour will be made, therefore, the superscripts f and t suffice to indicate fiber and transverse direction, respectively. Then, the plastic strain rates in longitudinal and transversal direction, normalized by the respective yield stress, are:

$$\frac{\dot{\underset{\sim}{\varepsilon}}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{f}}(\bar{\varepsilon}))}{y^{\mathrm{f}}} = \begin{bmatrix} 2\beta_{33} & 0 & 0 \\ 0 & \beta_{34} & 0 \\ 0 & 0 & \beta_{34} \end{bmatrix} \qquad \frac{\dot{\underset{\sim}{\varepsilon}}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{t}}(\bar{\varepsilon}))}{y^{\mathrm{t}}} = \begin{bmatrix} \beta_{34} & 0 & 0 \\ 0 & 2\beta_{44} & 0 \\ 0 & 0 & 2\beta_{44} - \beta_1 \end{bmatrix} \tag{21}$$

The three necessary equations can now be obtained from considering plastic Poisson's ratios for uniaxial loading in longitudinal direction (22_1) and transverse direction (22_2). The following index convention is used for the Poisson's ratios: Cause before effect, i.e. $\nu^p_{ce} = -\varepsilon^p_{ee}/\varepsilon^p_{cc}$.

$$\nu^p_{12} := -\frac{\varepsilon^p_{22}(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{f}})}{\varepsilon^p_{11}(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{f}})} = -\frac{\beta_{34}}{2\beta_{33}} \tag{22_1}$$

$$\nu_{21}^p := -\frac{\varepsilon_{11}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{t}})}{\varepsilon_{22}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{t}})} = -\frac{\beta_{34}}{2\beta_{44}} \qquad \nu_{23}^p := -\frac{\varepsilon_{33}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{t}})}{\varepsilon_{22}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{t}})} = +\frac{\beta_1}{2\beta_{44}} - 1 \tag{22$_2$}$$

Solving the resulting set of equations for $\beta_{i(j)}$ renders:

$$\beta_1 = 1 \qquad \beta_{33} = \frac{\nu_{21}^p}{2\nu_{12}^p\left(\nu_{23}^p+1\right)} \qquad \beta_{34} = -\frac{\nu_{21}^p}{\nu_{23}^p+1} \qquad \beta_{44} = \frac{1}{2\left(\nu_{23}^p+1\right)} \tag{23}$$

Specifying the plastic Poisson's ratios via (22) implies a certain amount of compressibility. For a general loading, the compressibility is obtained from the trace of (21) which simplifies to (24$_1$) for uniaxial longitudinal and (24$_2$) for uniaxial transversal loading. If an assumption for the compressibility is to be made any or both of these derived quantities can be substituted for the Poisson's ratios in (22).

$$\frac{\mathrm{tr}(\dot{\underset{\sim}{\varepsilon}}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{f}}))}{y^{\mathrm{f}}} = (2\beta_{33}+2\beta_{34}) \qquad = \frac{\nu_{21}^p(2\nu_{12}^p-1)}{\nu_{12}^p(1+\nu_{23}^p)} \tag{24$_1$}$$

$$\frac{\mathrm{tr}(\dot{\underset{\sim}{\varepsilon}}^p(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{t}}))}{y^{\mathrm{t}}} = (\beta_{33}+4\beta_{34}-\beta_1) = \frac{\nu_{21}^p+\nu_{23}^p-1}{(1+\nu_{23}^p)} \tag{24$_2$}$$

Clearly, the generic approach (19) allows for a lot of flexibility when it comes to specify model assumptions. If, for instance, not all 3 Poisson's ratios are available from experiments, assumptions regarding incompressibility or inextensibility, as appropriate for the given material, can provide the basis to determine the coefficients. E.g. it is apparent from (21) that plastic inextensibility could be achieved by $\beta_{33} = \beta_{34} = 0$. In that case, the single remaining coefficient β_{44} is to be determined from the last equation in (23). With the additional assumption of incompressibility $\beta_{44} = 1/4$ is obtained.

6 CONCLUSIONS

Considering materials with transversally isotropic symmetry, a systematic scheme to derive specialized forms of the constitutive functionals to model plasticity-like behaviour from a general isotropic tensor polynomial of order 2 was reviewed. Calibration procedures for both the yield surface and the plastic potential were discussed. In particular, the conditions leading to a yield surface independent from longitudinal (fiber) stress and hydrostatic stress were demonstrated, referring to an appropriate decomposition of the stress tensor in plasticity inducing and reactive parts. The adaption to different decompositions is straightforward.

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