Nonlinear Asymptotic Homogenization method for periodic elastic material with large deformation

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ABSTRACT

It is a great challenge to obtain the accurate stress and deform states of periodic materials for large number of microstructures are required for the multi scale calculation, Asymptotic Homogenization (AH) method, which has rigorous mathematical foundation and has been successfully used in prediction of equivalent modules, is extended to the multi-scale nonlinear deformation analysis in the paper. Numerical examples demonstrate the effectively of the Nonlinear Asymptotic Homogenization proposed in this paper.

1 INTRODUCTION

Hyperplastic material enables recoverable geometrically nonlinear deformation under finite stretches, which inspire a new way to drive deformation to realize the nonlinear constitutive relationship[1] and tunable functions[2], which can be used in the design of smart equipment such as comfortable wearable products, intelligent machines and tunable phononic metamaterial. As the programmable nonlinear mechanical properties and tunable functions are essentially dependent on the geometry, topology and material properties of the microstructure in periodic elastic material, many attention has been paid on the design of microstructure of periodic material with large deformation and the corresponding deformation operating ways[3]. The high expectations for programmable mechanical properties and tunable functions, such as tunable band-gap properties between different frequency ranges, require more accurate analysis and delicate tailored in the microsturcutre of the large-deformation periodic elastic material. It is a great challenge to obtain the accurate stress and deform states of periodic materials for large number of microstructures are required for the multi scale calculation, and representative volume element (RVE) method is the most widely used method in predicting effective mechanical properties in the analysis of large deformation periodic materials by now, however this method always brings complex boundary condition hard to deal with. Asymptotic Homogenization (AH) method, which has rigorous mathematical foundation and has been successfully used in prediction of equivalent modules and design optimization of linear deformation composite material [4], is extended to the multi-scale nonlinear deformation analysis in the paper. Different form linear problem, the modulus of periodic elastic material with large deformation is dependent on the deformation state in the microstructure, and the material modules are nonlinear related with deformation.

2 NONLINEAR ASYMPTOTIC HOMOGENIZATION FORMULATION

Through loading step by step, the expression of the effective modules for nonlinear geometric periodic material with none initial stress is the same with that of linear periodic material, expressed in Eq (1), and the characteristic displacement increment can be calculated in the same way by Eq (2).

$$ {}_t\boldsymbol{D}^H_{ijkl} = \frac{1}{|Y|}\int_Y \left({}_t\boldsymbol{D}_{ijkl} - {}_t\boldsymbol{D}_{ijpq}\frac{\partial\, {}_t\boldsymbol{\chi}^{pq}_k}{\partial y_l} \right) {}^t dy \qquad t=0 \tag{1} $$

$$ \frac{1}{|Y|}\int_Y \left({}_t\boldsymbol{D}_{ijkl} - {}_t\boldsymbol{D}_{ijpq}\frac{\partial\, {}_t\boldsymbol{\chi}^{pq}_k}{\partial y_l} \right) \frac{\partial v_i}{\partial y_j}\, {}^t dy = 0 \qquad t=0 \tag{2} $$

Where Y is the microstructure domain, ${}_t\boldsymbol{D}^H_{ijkl}$ is elastic modulus of material in the microstructure of time t,

and ${}_t\boldsymbol{\chi}_k^{pq}$ is characteristic displacement increment, and suppose the characteristic displacement increment at *t* time is expressed as:

$$ {}_t u_k^{(1)}(x,y) = -{}_t\boldsymbol{\chi}_k^{pq}(y)\frac{\partial_t u_p^{(0)}(x)}{\partial x_q} \tag{3} $$

When there is an initial stress in the geometrical nonlinear microstructure, with the assumption that the strain of the macro material for every step is the same, the deformation related nonlinear modules can be expressed as equation (4) based on Update Lagrange formulation.

$$ {}_t\boldsymbol{D}_{ijkl}^H = \sum_{t=0}^{t}\frac{1}{|Y|}\int_Y\left({}_t\boldsymbol{D}_{ijkl} - {}_t\boldsymbol{D}_{ijpq}\frac{\partial_t\chi_k^{pq}}{\partial y_l}\right){}^t dy + \frac{1}{|Y|}\int_Y\left({}^t\boldsymbol{\tau}_{ij}^{\varepsilon} - {}^t\boldsymbol{\tau}_{ij}^{\varepsilon}\frac{\partial_t\chi_k^{pq}}{\partial y_l}\right){}^t dy \quad t\ge 1 \tag{4} $$

Which the characteristic displacement increment for every load step can be solve the following equation

$$ \begin{aligned} &\left[\frac{1}{|Y|}\int_Y\left({}_t\boldsymbol{D}_{ijkl} - {}_t\boldsymbol{D}_{ijmn}\frac{\partial_t\chi_m^{kl}}{\partial y_n}\right)\frac{\partial v_i}{\partial y_j}{}^t dy\right]\frac{\partial_t u_k^{(0)}}{\partial x_l} + \left[\left(\frac{1}{|Y|}\int_Y{}^t\boldsymbol{\tau}_{ij}\frac{\partial v_k}{\partial y_j}{}^t dy\right) - \left(\frac{1}{|Y|}\int_Y{}^t\boldsymbol{\tau}_{nj}\frac{\partial_t\chi_m^{ki}}{\partial y_n}\frac{\partial v_m}{\partial y_j}{}^t dy\right)\right]\frac{\partial_t u_k^{(0)}}{\partial x_i} \\ &= -\frac{1}{|Y|}\int_Y{}^t\boldsymbol{\tau}_{ij}\frac{\partial v_i}{\partial y_j}{}^t dy, \qquad \forall v(x)\in \mathrm{V}_{\Omega^\varepsilon} \quad t\ge 1 \end{aligned} \tag{5} $$

Where ${}_t\boldsymbol{D}$ is the constitutive matrix for time *t*, and ${}^t\boldsymbol{\tau}$ is the stress matrix for time *t*. According to the above formulation, the effective elastic modulus for time *t* can be obtained.

As the effective mechanical properties of the periodic geometrically nonlinear material are dependent on the deformation state of the material, thus the macrostructural strain of material should be set privously. The macro-structural strain $\frac{\partial_t u_k^{(0)}}{\partial x_l}$ can be set as $\frac{\partial_t u_k^{(0)}}{\partial x_l} = \left\{\varepsilon_x^{(0)}, \varepsilon_y^{(0)}, \gamma_{xy}^{(0)}\right\}$. Then nonlinear element formulation to solve equation (5) can be expressed in equation (6) constrained with periodic boundary conditions.

$$ \begin{aligned} &\frac{1}{|Y|}\left[\left(\int_Y\{\boldsymbol{v}\}^T\left[{}_t\boldsymbol{B}_L\right]^T\left[{}_t\boldsymbol{D}\right]{}^t dY\right) - \left(\int_Y\{\boldsymbol{v}\}^T\left[{}_t\boldsymbol{B}_L\right]^T\left[{}_t\boldsymbol{D}\right]\left[{}_t\boldsymbol{B}_L\right]{}^t dY\right)\left[{}_t\boldsymbol{\chi}\right]\right]\left\{{}_t\boldsymbol{\varepsilon}^{(0)}\right\} \\ &+\frac{1}{|Y|}\left[\left(\int_Y\{\boldsymbol{v}\}^T\left[{}_t\boldsymbol{G}\right]^T\left[\boldsymbol{M}\right]{}^t dY\right) - \left(\int_Y\{\boldsymbol{v}\}^T\left[{}_t\boldsymbol{G}\right]^T\left[\boldsymbol{M}\right]\left[{}_t\boldsymbol{G}\right]{}^t dY\right)\left[{}_t\boldsymbol{\chi}\right]\right]\left\{{}_t\boldsymbol{\varepsilon}^{(0)}\right\} \\ &= -\frac{1}{|Y|}\left(\int_Y\{\boldsymbol{v}\}^T\left[{}_t\boldsymbol{B}_L\right]^T\left[{}^t\boldsymbol{\tau}\right]{}^t dY\right), \qquad \forall v(x)\in \mathrm{V}_{\Omega^\varepsilon} \end{aligned} \tag{6} $$

Which can be written as:

$$ \left(\left[{}_t\boldsymbol{K}_L\right] + \left[{}_t\boldsymbol{K}_{NL}\right]\right)\left[{}_t\boldsymbol{\chi}\right]\left\{{}_t\boldsymbol{\varepsilon}^{(0)}\right\} = \left(\left[{}_t\boldsymbol{F}_L^{kl}\right] + \left[{}_t\boldsymbol{F}_{NL}^{kl}\right]\right)\left\{{}_t\boldsymbol{\varepsilon}^{(0)}\right\} + \left[{}_t^{\tau}\boldsymbol{F}\right], \qquad \forall v(x)\in \mathrm{V}_{\Omega^\varepsilon} \tag{7} $$

Where the linear stiffness matrix and nonlinear stiffness matrix are found from

$$ {}_t\boldsymbol{K}_L = {}_t\boldsymbol{B}_L^T\,{}_t\boldsymbol{D}\,{}_t\boldsymbol{B}_L, \quad {}_t\boldsymbol{K}_{NL} = {}_t\boldsymbol{B}_{NL}^T\,{}^t\boldsymbol{\tau}\,{}_t\boldsymbol{B}_{NL} \tag{8} $$

And the linear force vector and nonlinear force vector are found from

$$ {}_t\boldsymbol{f}_L^{kl} = {}_t\boldsymbol{B}_L^T\,{}_t\boldsymbol{D}, \quad {}_t\boldsymbol{f}_{NL}^{kl} = {}_t\boldsymbol{B}_{NL}^T\,{}^t\boldsymbol{\tau} \tag{9} $$

Where ${}_t\boldsymbol{D}$ is the constitutive matrix for time *t*, ${}_t\boldsymbol{B}_L$ and ${}_t\boldsymbol{B}_{NL}$ are the linear finite element strain-displacement matrix and nonlinear finite element strain-displacement matrix for time *t*, and ${}^t\boldsymbol{\tau}$ is the stress matrix for time *t*.

The flowchart for Nonlinear Asymptotic Homogenization method proposed in this paper is shown in Figure 1, and arc-length method is used to solve the nonlinear equilibrium equations in the microstructure for each load step.

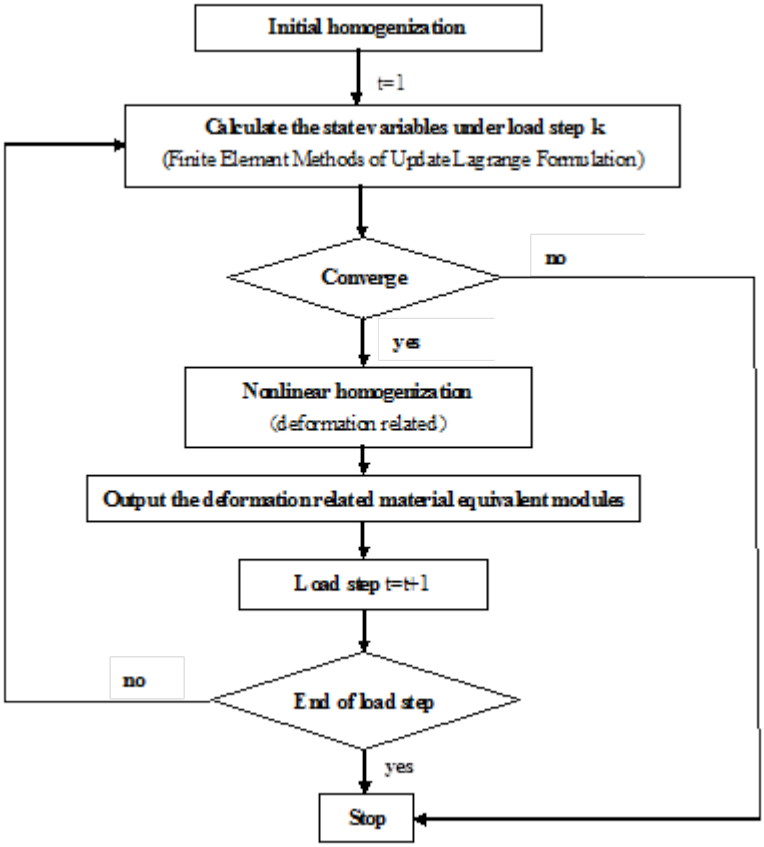

Figure 1: Flowchart for Nonlinear Homogenization method

3 NUMERICAL EXAMPLE

In this example, the periodic material consists of a base unit cell containing 4 circular holes, which is shown in Fig. 2. The size of the base cell is 20cm×20cm, and radiuses of the circles are 3.5cm and 4.9cm respectively. The properties of the solid material are E=220kPa, μ=0.5. The effective mechanical properties of the periodic material with compressive strain in y direction press and no strain in x direction is investigated.

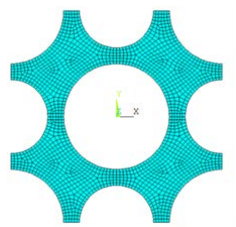

Figure 2: unit base cell

3.1 Numerical results of Nonlinear Asymptotic Homogenization Method

The proposed NAH method can used to predict the effective stress and strain relation, and in this example, the macro-structural strain is simply set as $\{0, \varepsilon_y^{(0)}, 0\}$, which mean that there is only macro-strain in y direction. The macro-strain is set as -0.001 for each step, to strain range between 0 to 0.1 is simulate by integrating the NAH method for 100 times. And the result of the equivalent nonlinear stress and strain relation predicted by NAH method shown in Figure 3.

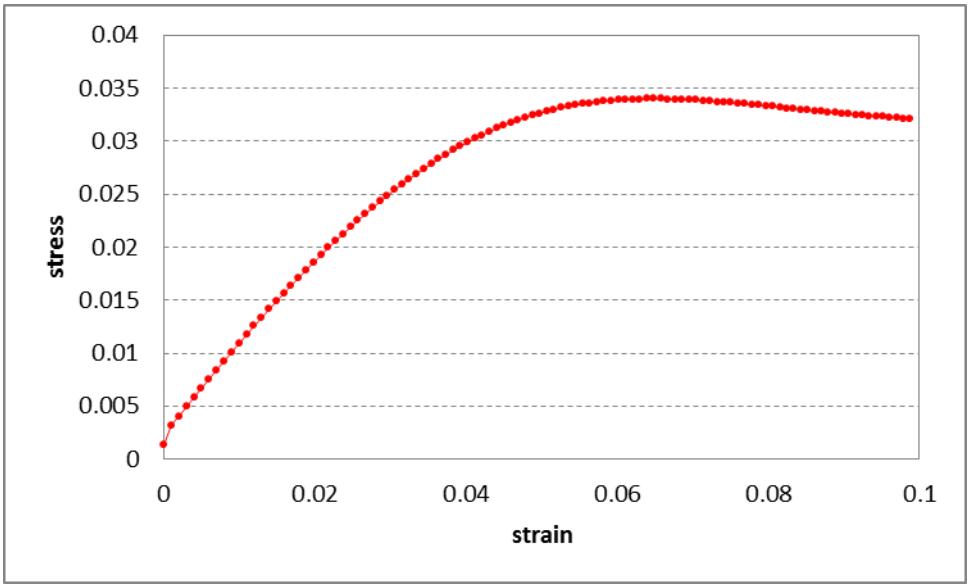

Figure 3: the equivalent nonlinear stress and strain relation predicted by NAH method

3.2 Numerical results of periodic material with specific deformation states

In order to demonstrate the effectiveness of the proposed NAH method proposed in this paper, a periodic porous structure under periodic condition between the left side and right side is introduced to realized the macro-structural deformation state of $\{0, \varepsilon_y^{(0)}, 0\}$.The corresponding deformation state and stress for different unidirectional strains are shown in Figure 4, and the comparison of the nonlinear stress-strain curve of the periodic geometrically nonlinear porous structure and the equivalent stress-strain curve obtained by NAH method is shown in Figure 5.

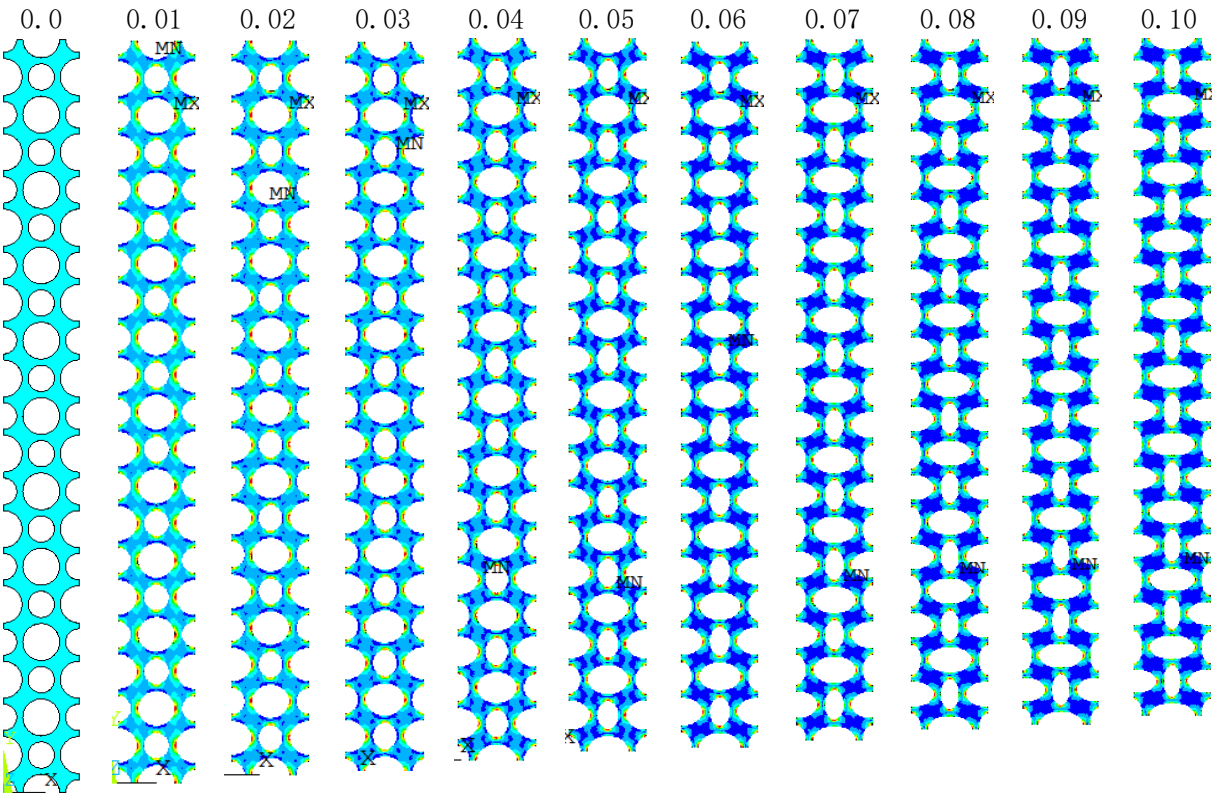

Figure 4: the deformation state and stress for 0-0.1 macro- strain in y direction

3.3 Discussions

The trend of the nonlinear curve between simulation of periodic geometrically nonlinear structure and the equivalence by the proposed NAH method is consistent well, the effectiveness of the NAH method proposed in demonstrated by this numerical example, the numerical errors come from both the mesh in pre-treatment and the step-length of strain in deformation simulation should be considered. Furthermore, this FEM-based NAH method is efficient for various types of periodic material.

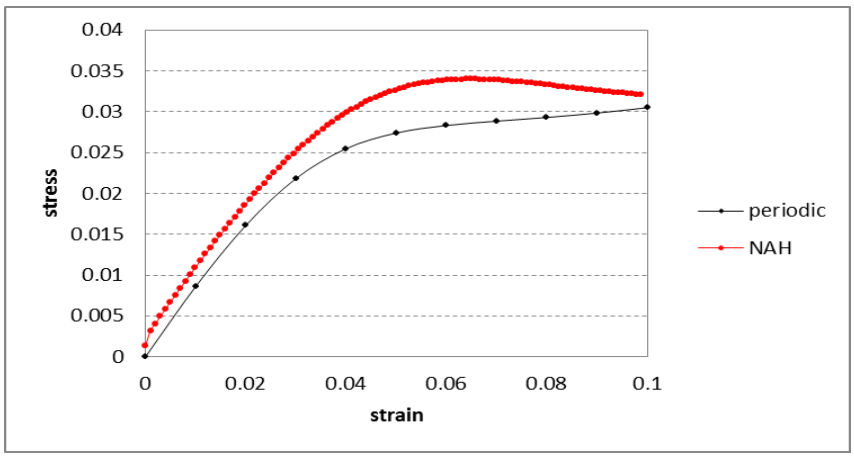

Figure 5: The comparison of nonlinear stress-strain curves between the simulation of periodic structure and the equivalence by NAH method

4 CONCLUSIONS

The proposed Nonlinear Asymptotic Homogenization (NAH) method extend the traditional homogenization theory to predict the effective nonlinear mechanical properties and calculate the multi-scale large deform and stress state for periodic elastic material subjected to step by step load, which can effectively improve the computational efficiency in the analysis and design of periodic elastic material with programmable mechanical properties and tunable functions.

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