1 Introduction

Multilayered structures made of piezoelectric and piezomagnetic materials have been used widely due to their special properties of converting the mechanical energy into electrical or magnetic energy and vice versa [1-3]. For the piezoelectric materials, various studies have been carried out to analyze and to design such multilayered smart structures. Furthermore, for the accurate prediction of static and dynamic behaviors, various coupled thermo-electro-elastic analysis containing thermal effects that are significant in multiphysics problems have been carried out. Elasticity solution has been proposed [4] and higher-order zigzag models have been reported [5,6].

In the same context, the analysis of magneto-electro-elastic (MEE) materials has been increasingly demanded recently due to their unique characteristics. For the analysis of the multilayered rectangular MEE plates, Pan [2] obtained the analytical solution under the static sinusoidal load based on the quasi-Stroh formalism and the propagator matrix method. Moreover, Pan and Heyliger [3] solved the cylindrical bending problem of MEE plates with simply-supported edge condition. The vibration analysis of MEE plates has also been carried out using a layerwise-type approximation by Ramirez et al. [7]. In addition, Annigeri et al. [8] studied the free vibration behavior of MEE beam based on the membrane-type finite element model. However, even though the previous analyses has been reported for engineering applications, one of the major drawbacks of them is that the number of unknowns is dependent upon the number of layers, which means those models are not efficient and thus they have limitations to be applied to the large-scale, sensing and actuating problem (i.e., dynamic analysis of the multilayered, coupled MEE plates). Thus, more efficient theory which also contains the accuracy are required. Among the various studies of the multilayered plate structures, an efficient higher order plate theory (EHOPT) proposed by Cho and Parmeter [9] is the best performer in displacement-based zigzag theories [10] and recommended for the analysis of the MEE plates since numerous studies have been verified the accuracy and efficiency of the EHOPT by analyzing the fully coupled piezoelectric composite plates [7] as well as the conventional composite laminates. This theory reduces the known variables using the top/bottom boundary conditions and the transverse direction flux continuity conditions. In this study, the multilayered MEE plates are considered to carry out the fully coupled magneto-electro-elastic analysis built upon the EHOPT. The displacement field, electric potential and magnetic potential are assumed as a third order zigzag functions. The number of unknowns is reduced effectively by applying the top/bottom conditions and transverse direction flux continuity conditions. For the practical usage of the present method, finite element discretization based on the beam-type model is applied. To investigate the different responses of the elastic, electric and...
magnetic quantities, various layups and different loadings are considered by comparing them with the results reported in the literatures.

2 Formulation

A three-dimensional multilayered magneto-electro-elastic plate is shown in Fig. 1. The constitutive equations for a linear, anisotropic and magneto-electro-elastic solid can be written as [1-3]

\begin{align*}
\sigma_{ij} &= C_{ijkl} \varepsilon_{kl} - e_{ijk} E_k - q_{ijk} H_k \\
D_i &= e_{ilk} \varepsilon_{kl} + \gamma_{ik} E_k + d_{ik} H_k \\
B_i &= q_{ilk} \varepsilon_{kl} + d_{ik} E_k + \mu_{ik} H_k 
\end{align*}

where \(\sigma_{ij}\), \(D_i\) and \(B_i\) are the stress, electric displacement and magnetic induction, respectively. \(\varepsilon_{kl}\), \(E_k\) and \(H_k\) are the strain, electric field and magnetic field, respectively. \(C_{ijkl}\), \(\gamma_{ik}\) and \(\mu_{ik}\) are the elastic, dielectric and magnetic permeability coefficients, respectively. \(e_{ijk}\), \(q_{ijk}\) and \(d_{ik}\) are the piezoelectric, piezomagnetic and magnetoelectric coefficients, respectively. The piezoelectric and magnetostrictive layers are made of BaTiO\(_3\) and CoFe\(_2\)O\(_4\), respectively. Corresponding equilibrium equations can be written as

\begin{align*}
\sigma_{ij,j} &= 0, \quad D_{i,i} = 0, \quad B_{i,i} = 0
\end{align*}

The strain-displacement, electric potential and magnetic potential relationship can be expressed as follows:

\begin{align*}
\varepsilon_{ij} &= \frac{1}{2} (u_{i,j} + u_{j,i}), \quad E_i = -\phi_i, \quad H_i = -\psi_i
\end{align*}

In this study, a fully coupled higher-order zigzag theory is proposed for efficient modeling. The displacement field is assumed by superimposing zigzag linear field to the globally varying field as follows:

\begin{align*}
u_i(x, z) &= u^0_i(x) + \chi_i(x)z + \xi_i(x)z^2 \\
&\quad + \eta_i(x)z^3 + \sum_{k=1}^{N-1} S^i_k (z - z_k) \mathcal{H}(z - z_k) \\
u_j(x, z) &= w(x) + \tau_j(x)z + \zeta_j(x)z^2 \\
&\quad + \sum_{k=1}^{N-1} r_{sk}^j (x)(z - z_k) \mathcal{H}(z - z_k)
\end{align*}

where \(\mathcal{H}(z-z_k)\) is a Heaviside unit step function. \(u^0_i\) and \(w\) represent the in-plane displacement and the out-of-plane displacement from the reference plane, respectively. \(\chi_i\) are the rotation of the normal about \(x_1\) axis. For the in-plane displacement, a cubic varying field is applied. Whereas, for the out-of-plane displacement, a quadratic field is assumed in order to express the physical behavior of the displacement field accurately. In Eq. (4), the beam-type model which contains \(x, z\) and no \(y\) axis is considered for the simplicity of the problem. Similar to the construction of the displacement field, the electric and magnetic potential are obtained by superimposing linear zigzag field onto the global cubic smooth field as follows:

\begin{align*}
\phi(x, z) &= \phi_0(x) + \phi_1(x)z + \phi_2(x)z^2 \\
&\quad + \phi_3(x)z^3 + \sum_{k=1}^{N-1} \phi_{sk}^i (x)(z - z_k) \mathcal{H}(z - z_k) \\
\psi(x, z) &= \psi_0(x) + \psi_1(x)z + \psi_2(x)z^2 \\
&\quad + \psi_3(x)z^3 + \sum_{k=1}^{N-1} \psi_{sk}^i (x)(z - z_k) \mathcal{H}(z - z_k)
\end{align*}

The unknown variables in Eqs. (4-6) can be reduced by applying boundary conditions of top/bottom surface and introducing the continuity conditions of transverse stresses, transverse electric and magnetic displacement between each layer. The top/bottom boundary conditions can be expressed as
EFFICIENT HIGHER ORDER ZIG-ZAG THEORY FOR COUPLED MAGNETO-ELECTRO-ELASTIC COMPOSITE LAMINATES

\[
\begin{align*}
\sigma_3|_{z=0} &= \sigma_3|_{z=h} = 0 \\
\sigma_3|_{z=0} &= \overline{p}_0^{(0)}, \quad \sigma_3|_{z=h} = \overline{p}_0^{(h)} \\
\phi|_{z=0} &= \phi^{(0)}, \quad \phi|_{z=h} = \phi^{(h)} \\
\psi|_{z=0} &= \psi^{(0)}, \quad \psi|_{z=h} = \psi^{(h)} \\
\end{align*}
\] (7)

where \( h \) is a thickness of the multilayered plate. For the simplicity, engineering stress and strain notations are applied. The continuity conditions are applied as follows:

\[
\sigma_3|_{z=0} = \sigma_3|_{z=h}, \quad \sigma_5|_{z=0} = \sigma_5|_{z=h}, \quad D_3|_{z=0} = D_3|_{z=h}, \quad B_3|_{z=0} = B_3|_{z=h}
\] (8)

It must be noted that the stress and displacement fields which obtained from Eqs. (2-8) are statically admissible stress field and kinematically admissible displacement field, respectively, which indicates the fields satisfy proper thickness direction static and kinematic conditions. The equations can only be solved via the eigenvalue problem using the exponential functions. Thus, in this study, by ignoring the in-plane strain contribution to the transverse stress, transverse electric displacement and transverse magnetic induction in the constitutive equations, one can solve the Eqs. (2-8) without using the exponential functions and deriving the eigenvalue problem. The reduced constitutive equations of \( k \)-th layer are as follows, which only used in applying the continuity conditions.

From the Eqs. (3-9), the layer dependent variables, \( S_{1k} \), \( \phi^{sk} \) and \( \psi^{sk} \), are determined in terms of the primary variables of the reference plane as follows:

\[
\begin{align*}
\sigma_3^{(k)} &= C_{33}^{(k)} \varepsilon_3^{(k)} - C_{33}^{(k)} e_3^{(k)} - q_{33}^{(k)} H_3^{(k)} \\
\sigma_5^{(k)} &= C_{55}^{(k)} 2 \varepsilon_5^{(k)} - C_{55}^{(k)} E_5^{(k)} - q_{55}^{(k)} H_5^{(k)} \\
D_3^{(k)} &= e_{33}^{(k)} \varepsilon_3^{(k)} + \gamma_{33}^{(k)} E_3^{(k)} + d_{33}^{(k)} H_3^{(k)} \\
B_3^{(k)} &= q_{33}^{(k)} e_3^{(k)} + d_{33}^{(k)} E_3^{(k)} + \mu_{33}^{(k)} H_3^{(k)}
\end{align*}
\] (9)

From the Eqs. (3-9), the layer dependent variables, \( S_{1k} \), \( \phi^{sk} \) and \( \psi^{sk} \), are determined in terms of the primary variables of the reference plane as follows:

\[
\begin{align*}
\varepsilon_3^{(k)} &= K_f^{(k)} \cdot f + K_{11}^{(k)} \phi_2 + K_{11}^{(k)} \phi_3 \\
\phi^{sk} &= M_f^{(k)} \cdot f + M_1^{(k)} \phi_2 + M_1^{(k)} \phi_3 \\
\psi^{sk} &= P_f^{(k)} \cdot f + P_1^{(k)} \psi_2 + P_1^{(k)} \psi_3
\end{align*}
\] (10)

where the coefficients of each primary variable are omitted for the limited space. The vector which contains the top/bottom boundary condition is expressed as,

\[
\begin{align*}
\begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} &= \begin{pmatrix} \bar{p}_0^{(0)} & \bar{p}_0^{(h)} \\ \bar{\phi}^{(0)} & \bar{\phi}^{(h)} \\ \bar{\psi}^{(0)} & \bar{\psi}^{(h)} \end{pmatrix}
\end{align*}
\] (11)

The number of the primary variables also reduces by applying Eq. (7) to the Eqs. (4-6) and Eq. (10).

3 Virtual Work Principle and Constitutive Equations

The virtual work for a three-dimensional magneto-electro-elastic problem is stated as

\[
\delta W = \int_{\Omega} \left( \sigma_1 \delta \varepsilon_1 + \sigma_2 \delta \varepsilon_2 + \sigma_3 \delta \varepsilon_3 - D_3 \delta E_3 \\
- D_3 \delta E_3 - B_3 \delta H_1 - B_3 \delta H_3 \right) dV
\] (12)
where \( \bar{p}_i \) denotes the in-plane traction which did not consider in Eq. (7). The stress resultants can be obtained by integrating the stresses in the through-the-thickness direction as shown in Eq. (13). The constitutive equations of the laminate can be derived using the Eqs. (3-6) and Eqs. (11-12) as follows:

\[
\begin{align*}
\left[ R_1^{(0)} \right] & = \int_0^h \sigma_y \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ R_3^{(0)} \right] & = \int_0^h \sigma_z \left[ 1 \quad z \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ V_1^{(0)} \right] & = \int_0^h \sigma_z \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ V_3^{(0)} \right] & = \int_0^h \sigma_z \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ F_1^{(0)} \right] & = \int_0^h D_1 \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ F_3^{(0)} \right] & = \int_0^h D_5 \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ G_i^{(0)} \right] & = \int_0^h B_i \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ G_i^{(k)} \right] & = \int_0^h B_i \left[ 1 \quad z \quad z^2 \quad \mathcal{H}(z-z_k) \right] dz
\end{align*}
\] (13)

\[
\begin{align*}
\left[ R_1^{(1)} \right] & = \int_0^h \sigma_y \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ R_3^{(1)} \right] & = \int_0^h \sigma_z \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ V_1^{(1)} \right] & = \int_0^h \sigma_z \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ V_3^{(1)} \right] & = \int_0^h \sigma_z \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ F_1^{(1)} \right] & = \int_0^h D_1 \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ F_3^{(1)} \right] & = \int_0^h D_5 \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ G_i^{(1)} \right] & = \int_0^h B_i \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz \\
\left[ G_i^{(k)} \right] & = \int_0^h B_i \left[ 1 \quad z \quad z^2 \quad z^3 \quad \mathcal{H}(z-z_k) \right] dz
\end{align*}
\] (14)

The primary displacement, magnetic and electric potential unknowns are expressed in terms of nodal values and interpolation functions as follows:

\[
\left[ \begin{array}{c}
\bar{u}_i^0 \\
\bar{\eta}_i
\end{array} \right] = \sum_{m=1}^n N_m \left[ \begin{array}{c}
\bar{u}_i^0 \\
\bar{\eta}_i
\end{array} \right] m,
\]

\[
\begin{align*}
\left[ \begin{array}{c}
w \\
\phi_i \\
\psi_i \\
\phi_i
\end{array} \right] & = \sum_{m=1}^n Y_m \left[ \begin{array}{c}
w \\
\phi_i \\
\psi_i \\
\phi_i
\end{array} \right] m \\
& = \sum_{m=1}^n Q_m \left[ \begin{array}{c}
w \\
\phi_i \\
\psi_i \\
\psi_i
\end{array} \right] m
\end{align*}
\] (16)

where \( n \) is the number of nodes in a typical finite element. \( N_m \) is a Lagrangian interpolation function and \( Q_m \) are Hermite interpolation functions. The components of the field variables and the nodal displacement vectors are given by

\[
\begin{align*}
\mathbf{e}_1 & = \left\{ e_1^{(0)} \quad e_1^{(1)} \quad e_1^{(2)} \quad e_1^{(1)} \quad e_1^{(3)} \quad \cdots \quad e_1^{N-1} \right\}^T \\
\mathbf{e}_3 & = \left\{ e_3^{(0)} \quad e_3^{(1)} \quad \cdots \quad e_3^{N-1} \right\}^T \\
\mathbf{e}_4 & = \left\{ e_4^{(0)} \quad e_4^{(1)} \quad \cdots \quad e_4^{N-1} \right\}^T \\
\mathbf{e}_5 & = \left\{ e_5^{(1)} \quad \cdots \quad e_5^{N-1} \right\}^T \\
\mathbf{e}_7 & = \left\{ e_7^{(0)} \quad e_7^{(1)} \quad \cdots \quad e_7^{N-1} \right\}^T \\
\mathbf{e}_9 & = \left\{ e_9^{(0)} \quad e_9^{(1)} \quad \cdots \quad e_9^{N-1} \right\}^T \\
\mathbf{e}_11 & = \left\{ e_{11}^{(0)} \quad e_{11}^{(1)} \quad e_{11}^{(2)} \quad e_{11}^{(3)} \quad \cdots \quad e_{11}^{N-1} \right\}^T \\
\mathbf{e}_{13} & = \left\{ e_{13}^{(0)} \quad e_{13}^{(1)} \quad \cdots \quad e_{13}^{N-1} \right\}^T \\
\mathbf{e}_{15} & = \left\{ e_{15}^{(0)} \quad e_{15}^{(1)} \quad e_{15}^{(2)} \quad e_{15}^{(3)} \quad \cdots \quad e_{15}^{N-1} \right\}^T \\
\mathbf{e}_{17} & = \left\{ e_{17}^{(0)} \quad e_{17}^{(1)} \quad e_{17}^{(2)} \quad e_{17}^{(3)} \quad \cdots \quad e_{17}^{N-1} \right\}^T \\
\mathbf{e}_{19} & = \left\{ e_{19}^{(0)} \quad e_{19}^{(1)} \quad e_{19}^{(2)} \quad e_{19}^{(3)} \quad \cdots \quad e_{19}^{N-1} \right\}^T \\
\mathbf{e}_{21} & = \left\{ e_{21}^{(0)} \quad e_{21}^{(1)} \quad e_{21}^{(2)} \quad e_{21}^{(3)} \quad \cdots \quad e_{21}^{N-1} \right\}^T \\
\mathbf{e}_{23} & = \left\{ e_{23}^{(0)} \quad e_{23}^{(1)} \quad e_{23}^{(2)} \quad e_{23}^{(3)} \quad \cdots \quad e_{23}^{N-1} \right\}^T
\end{align*}
\] (17)

4 Finite Element Model

To assess the validity of the present method, a finite element is developed for one-dimensional problems.
where \((k)\), the superscript with brackets, is the coefficient of \(z^k\). \(N\) denotes the number of layers. \([B]\) represents the strain-displacement matrix, and the details of the matrix are omitted for the limited space. The stiffness matrix can be constructed from the virtual work principle as follows:

\[
\delta W = \int \epsilon^T C \delta \epsilon dx
\]

\[
= \left( \int \left\{ u_n \right\}^T \left[ B \right]^T \left[ B \right] dx \right) \delta \left\{ u_n \right\} - \left( \int \epsilon^T C \left[ B \right] dx \right) \delta \left\{ u_n \right\}
\]

\[
= \left( \left[ K \right] \left\{ u_n \right\} - \left\{ f \right\} \right) \delta \left\{ u_n \right\} = 0
\]  

For the multilayered MEE laminates, the top/bottom boundary conditions are the external forces in the finite element problem. Finally, nodal unknowns can be obtained by solving the Eq. (19).

5 Conclusions

A fundamental higher order zigzag theory of multilayered MEE plate is presented in this paper. Starting from the fully-coupled three-dimensional field equations, the displacements and potentials are assumed as cubic and quadratic zigzag functions. The top/bottom boundary conditions and the continuity conditions are employed in order to reduce the layer-dependent variables. For the application of the present method, a finite element is developed for one-dimensional problems. The final finite element governing equations contain the fully coupled magneto-electro-elastic behaviors. To examine the coupling effects of MEE plate, the electrical and magnetic loadings as well as the mechanical loading are considered for various layup configurations. The present method can be used as an efficient tool for the analysis of such smart composite structures.

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